

# On the Cauchy problem for non-local Ornstein–Uhlenbeck operators

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**Abstract:** We study the Cauchy problem involving non-local Ornstein-Uhlenbeck operators in finite and infinite dimensions. We prove classical solvability without requiring that the Lévy measure corresponding to the large jumps part has a first finite moment. Moreover, we determine a core of regular functions which is invariant for the associated transition Markov semigroup. Such a core allows to characterize the marginal laws of the Ornstein-Uhlenbeck stochastic process as unique solutions to Fokker-Planck-Kolmogorov equations for measures.

**Keywords:** Ornstein–Uhlenbeck non-local operators; Cauchy problem; Lévy processes; core for Markov semigroups.

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## 1 Introduction and notation

In this paper we investigate solvability of the Cauchy problem involving non-local Ornstein-Uhlenbeck operators both in finite and infinite dimensions. We also determine a core of regular functions which is invariant for the transition Ornstein-Uhlenbeck semigroup. Differently with respect to recent papers (see [Ap07, Section 5], [Kn11, Section 4.1] and [We13, Section 2]) to study the core problem we do not require that the associated Lévy measure  $\nu$  corresponding to the large jumps part has a first finite moment (see (1.8)).

Let us first introduce the Ornstein-Uhlenbeck operator  $\mathcal{L}_0$  in  $\mathbb{R}^d$  and its asso-

ciated stochastic process. The operator  $\mathcal{L}_0$  is defined as

$$\begin{aligned} \mathcal{L}_0 f(x) &= \frac{1}{2} \sum_{j,k=1}^d Q_{jk} \partial_{x_j x_k}^2 f(x) + \sum_{j=1}^d a_j \partial_{x_j} f(x) + \sum_{j,k=1}^d A_{jk} x_k \partial_{x_j} f(x) \quad (1.1) \\ &+ \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - \mathbf{1}_{\{|y| \leq 1\}}(y) \sum_{j=1}^d y_j \partial_{x_j} f(x) \right) \nu(dy), \quad x \in \mathbb{R}^d, \end{aligned}$$

where  $\mathbf{1}_{\{|y| \leq 1\}}$  is the indicator function of the closed ball with center 0 and radius 1,  $Q = (Q_{ij})$  and  $A = (A_{ij})$  are given  $d \times d$  real matrices ( $Q$  being symmetric and non-negative definite). Moreover  $a = (a_1, \dots, a_d) \in \mathbb{R}^d$  and  $\nu$  is a Lévy jump measure, i.e.,  $\nu$  is a  $\sigma$ -finite Borel measure on  $\mathbb{R}^d$  such that

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty \quad (1.2)$$

( $a \wedge b$  indicates the minimum between  $a$  and  $b \in \mathbb{R}$ ). The function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  belongs to  $C_b^2(\mathbb{R}^d)$  (i.e.,  $f$  is bounded and continuous together with its first and second partial derivatives) and the integral in (1.1) is well defined thanks to the Taylor formula. The associated Ornstein-Uhlenbeck process (OU process) solves the following SDE driven by a Lévy process  $Z$ :

$$\begin{cases} dX_t = AX_t dt + dZ_t, & t \geq 0 \\ X_0 = x, & x \in \mathbb{R}^d \end{cases} \quad (1.3)$$

(see, for instance, [SY84], [SWYY96] and [Ma04]). The matrix  $A$  is the same as in (1.1) and  $Z = (Z_t)_{t \geq 0} = (Z_t)$  is a  $d$ -dimensional Lévy process uniquely determined in law by the previous  $Q$ ,  $a$  and  $\nu$  (cf. [Sa99, Section 9]). Ornstein-Uhlenbeck processes with jumps  $X = (X_t) = (X_t^x)$  have several applications to Mathematical Finance and Physics (see for instance, [BNS01], [CT04] and [GO00]). The corresponding transition Markov semigroup  $(P_t) = (P_t)_{t \geq 0}$  is called the Ornstein-Uhlenbeck semigroup (or Mehler semigroup):

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad t \geq 0, \quad (1.4)$$

for any  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  which is Borel and bounded (see also (2.5)). In Section 3.1 we prove well-posedness of the Cauchy problem

$$\begin{cases} \partial_t u(t, x) = \mathcal{L}_0 u(t, x) \\ u(0, x) = f(x), \quad x \in \mathbb{R}^d, \quad f \in C_b^2(\mathbb{R}^d), \end{cases} \quad (1.5)$$

where  $\mathcal{L}_0 u(t, x) = (\mathcal{L}_0 u(t, \cdot))(x)$  (see Theorem 3.3). We show that there exists a unique bounded classical solution given by  $u(t, x) = P_t f(x)$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^d$ . Our result is not covered by regularity results on singular pseudodifferential operators (cf. [Ko89]). Moreover, it can not be deduced by perturbation arguments using known results for the Ornstein-Uhlenbeck semigroup (see, in particular, [SY84] and [Ma04]). To prove solvability of (1.5) we first establish the crucial formula

$$P_t f(x) = f(x) + \int_0^t \mathcal{L}_0(P_s f)(x) ds, \quad t \geq 0, x \in \mathbb{R}^d, \quad f \in C_b^2(\mathbb{R}^d). \quad (1.6)$$

(see Theorem 3.1). Note that in [SY84, Theorem 3.1] it is proved that

$$P_t f(x) = f(x) + \int_0^t P_s(\mathcal{L}_0 f)(x) ds, \quad t \geq 0, x \in \mathbb{R}^d, \quad f \in C_K^2(\mathbb{R}^d). \quad (1.7)$$

(we write  $f \in C_K^2(\mathbb{R}^d)$  if  $f \in C_b^2(\mathbb{R}^d)$  and  $f$  has compact support). Even assuming  $f \in C_K^2(\mathbb{R}^d)$ , formula (1.6) can not be obtained directly from (1.7) since the space  $C_K^2(\mathbb{R}^d)$  is not invariant for the OU semigroup  $(P_t)$  (cf. Remark 3.6). On the other hand (1.7) does not hold in general for  $f \in C_b^2(\mathbb{R}^d)$  since  $\mathcal{L}_0 f$  can grow linearly and so  $P_t(\mathcal{L}_0 f)(x)$  could be not well-defined without requiring the additional assumption (cf. (2.2))

$$\int_{\{|y|>1\}} |y| \nu(dy) < \infty. \quad (1.8)$$

It is well-known that the OU semigroup  $(P_t)$  is a  $C_0$ -semigroup (or strongly continuous semigroup) of contractions on  $C_0(\mathbb{R}^d)$  (the Banach space of all real continuous functions on  $\mathbb{R}^d$  which vanish at infinity, endowed with the supremum norm); see [SY84], [Ma04] and [Tr12] for a more direct proof. Let us denote by  $\mathcal{L}$  its generator. Using (1.6) in Theorem 3.4 we show that

$$\mathcal{D}_0 = \left\{ f \in C_0^2(\mathbb{R}^d) \text{ such that } \sum_{j,k=1}^d A_{jk} x_k \partial_{x_j} f \in C_0(\mathbb{R}^d) \right\} \quad (1.9)$$

is *invariant* for the OU semigroup ( $f \in C_0^2(\mathbb{R}^d)$  if  $f \in C_b^2(\mathbb{R}^d)$  and  $f$  and its first and second partial derivatives belong to  $C_0(\mathbb{R}^d)$ ). Note that this property implies that the mapping:  $x \mapsto Ax \cdot DP_t f(x)$  is bounded on  $\mathbb{R}^d$  when  $f \in \mathcal{D}_0$  without assuming (1.8). It turns out that  $\mathcal{D}_0$  is also a core for  $\mathcal{L}$  and  $\mathcal{L}f = \mathcal{L}_0 f$ ,  $f \in \mathcal{D}_0$ . Clearly, if  $f \in \mathcal{D}_0$  then both (1.6) and (1.7) hold (see Corollary 3.5).

Starting from Section 4, we extend the main results of Section 3 to infinite dimensions, replacing  $\mathbb{R}^d$  with a given real separable Hilbert space  $H$ . Infinite dimensional Ornstein-Uhlenbeck processes are solutions of linear stochastic evolution equations and are formally similar to (1.3); we assume that  $A$  is the generator of a  $C_0$ -semigroup on  $H$  and  $Z$  is an  $H$ -valued Lévy processes. Such processes allow to solve basic linear SPDEs (cf. [DZ92], [Da04], [PZ07] and the references therein). Ornstein-Uhlenbeck processes with jumps in infinite dimensions were first studied in [Ch87]. A more general approach to such processes using generalised Mehler semigroups has been initiated in [BRS96] (see also [FR00], [LR02], [PZ06], [Kn11], [We13] and Remark 5.12).

In Theorem 5.1 we extend formula (1.6) to infinite dimensions when  $f \in C_b^2(H)$  and  $x \in D(A)$  (i.e.,  $x$  belongs to the domain of  $A$ ). We use such formula to show existence and uniqueness of solutions for an infinite-dimensional Cauchy problem like (1.5) when  $\mathbb{R}^d$  is replaced by  $H$  (see Theorem 5.5); we assume that the initial datum  $f$  belongs to  $C_b^2(H)$  and that a compatibility condition between  $Df(x)$  and  $A$  is satisfied (see the definition of the space  $C_A^2(H)$  in (5.3)). This result of well-posedness seems to be new even for local infinite dimensional OU operators corresponding to the case when  $Z$  is a Wiener process.

To study an infinite dimensional OU semigroup  $(P_t)$  it is natural to consider it as acting in  $C_b(H)$  or  $UC_b(H)$  which are both invariant for the semigroup. Here  $C_b(H)$  (resp.  $UC_b(H)$ ) consists of all real bounded and continuous (resp. uniformly continuous) functions on  $H$ . Indeed  $C_0(H)$  which generalizes  $C_0(\mathbb{R}^d)$  is invariant for  $(P_t)$  only under quite restrictive assumptions (see page 91 of [Ap07]). On the other hand, it is well known that  $(P_t)$  is not strongly continuous neither on  $C_b(H)$  nor in  $UC_b(H)$  if we consider the sup-norm topology (see [Ce94], [CG95], [Pr99], [GK01], [Ku03] where possible approaches to study Markov semigroups in  $C_b(H)$  or  $UC_b(H)$  are proposed). In  $C_b(H)$  one can define an infinitesimal generator  $\mathcal{L}$  in a pointwise sense (see (5.15)) or in other equivalent ways (see Remark 5.6). This generator coincides with the one investigated in [Ap07].

In Section 5.2 we determine two natural pointwise cores ( $\pi$ -cores)  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  for  $\mathcal{L}$ . They are both invariant for  $(P_t)$  and further the restriction of  $\mathcal{L}$  to  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  coincides with  $\mathcal{L}_0$ . The  $\pi$ -core  $\tilde{\mathcal{D}}_0$  is a kind of infinite-dimensional generalization of  $\mathcal{D}_0$  given in (1.9). On the other hand  $\mathcal{D}_1$  is similar to the space introduced in [Ma08] when the Lévy noise is a Wiener process. The definition of  $\mathcal{D}_1$  is a bit involved but this space can also be used to study generalised Mehler semigroups (cf. Remark 5.12). We discuss in Remark 5.11 another possible core used in [Ap07] under the assumption  $\int_{\{|x|>1\}} |x|\nu(dx) < \infty$  (see also [GK01]).

We mention that cores of regular bounded functions are also useful to investigate the Ornstein-Uhlenbeck semigroups in  $L^p(\mu)$  with respect to an invariant measure  $\mu$  assuming that such a measure exists (see [Ch87], [LR02], [Kn11] and the references therein). Moreover, such cores allow to study Fokker-Planck-Kolmogorov equations for measures (see [BDR04], [Ma08], [BDPR11], [Kn11], [We13] and the references therein). On this respect, in Section 6, following [Ma08], we show that both  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  can be used to prove that the marginal laws of Ornstein-Uhlenbeck processes are the unique solutions to Fokker-Planck-Kolmogorov equations for measures  $(\gamma_t)$ , i.e.,

$$\begin{cases} \frac{d}{dt} \int_H f(x) \gamma_t(dx) = \int_H \mathcal{L}_0 f(x) \gamma_t(dx), & f \in \mathcal{D}, t \geq 0, \\ \gamma_0 = \delta_x, \end{cases}$$

$x \in H$ ; here the space  $\mathcal{D}$  can be  $\tilde{\mathcal{D}}_0$  or  $\mathcal{D}_1$  (see Theorem 6.2).

**Notation.** In  $\mathbb{R}^d$  by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  we indicate the usual inner product and the Euclidean norm,  $d \geq 1$ . By  $B_b(\mathbb{R}^d, \mathbb{R}^k)$ ,  $d, k \geq 1$ , we denote the Banach space of all Borel and bounded functions  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}^k$  endowed with norm  $\|f\|_0 = \sup_{x \in \mathbb{R}^d} |f(x)|$ . When  $\mathbb{R}^k = \mathbb{R}$  we set  $B_b(\mathbb{R}^d, \mathbb{R}) = B_b(\mathbb{R}^d)$ ; this convention will be used for other function spaces as well.  $C_b(\mathbb{R}^d, \mathbb{R}^k)$  is the closed subspace of  $B_b(\mathbb{R}^d, \mathbb{R}^k)$  of all bounded and continuous functions. Moreover,  $UC_b(\mathbb{R}^d, \mathbb{R}^k)$  is the closed subspace of all bounded and uniformly continuous functions. We say that  $f \in C_0(\mathbb{R}^d, \mathbb{R}^k)$  if  $f \in C_b(\mathbb{R}^d, \mathbb{R}^k)$  and  $f$  vanishes at infinity (i.e., for any  $\epsilon > 0$  there exists a bounded set  $A \subset \mathbb{R}^d$  such that  $|f(x)| < \epsilon$  if  $x \in \mathbb{R}^d \setminus A$ ).

For each integer  $n \geq 1$ , we say that  $g \in C_b^n(\mathbb{R}^d)$  if  $g \in C_b(\mathbb{R}^d)$  and  $g$  is  $n$ -times (Fréchet) differentiable on  $\mathbb{R}^d$  with all the (Fréchet) derivatives  $D^j g$  which are continuous and bounded on  $\mathbb{R}^d$ ,  $j = 1, \dots, n$ . We use  $C_0^2(\mathbb{R}^d)$  to denote the space of functions  $f \in C_b^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$  such that  $Df$  and  $D^2 f$  vanish at infinity. Finally,  $C_K^2(\mathbb{R}^d)$  denotes the space of functions  $f \in C_0^2(\mathbb{R}^d)$  with compact support.

Given a real separable Hilbert space  $H$  and a linear bounded operator  $T : H \rightarrow H$ , we denote by  $\|T\|_L$  its operator norm. If in addition  $T$  is an Hilbert-Schmidt operator then

$$\|T\|_{HS} = \left( \sum_{k \geq 1} |Te_k|^2 \right)^{1/2}$$

denotes its Hilbert-Schmidt norm (here  $(e_k)$  is an orthonormal basis in  $H$  and  $\langle \cdot, \cdot \rangle$ ,  $|\cdot|$  denote the inner product and the norm in  $H$ ). We indicate by  $L_2(H)$  the space consisting of all Hilbert-Schmidt operators from  $H$  into  $H$ ; it is a real separable Hilbert space endowed with the inner product:  $T \cdot S = \text{Tr}(T^*S) = \sum_{k \geq 1} \langle Te_k, Se_k \rangle$ ,  $S, T \in L_2(H)$  (we refer to Appendix C in [DZ92] for more details).

The previous function spaces can be easily generalised when  $\mathbb{R}^d$  is replaced by  $H$ , i.e., we can consider the spaces  $B_b(H)$ ,  $C_b(H)$ ,  $UC_b(H)$ ,  $C_0(H)$ ,  $C_b^k(H)$ ,  $k = 1, 2$ . In particular,  $f \in C_b^1(H)$  if  $f : H \rightarrow \mathbb{R}$  is Fréchet differentiable in  $H$  and  $f : H \rightarrow \mathbb{R}$ ,  $Df : H \rightarrow H$  are bounded and continuous (we write  $Df \in C_b(H, H)$ ). Moreover,  $f \in C_b^2(H)$  if  $f \in C_b^1(H)$ ,  $f : H \rightarrow \mathbb{R}$  is twice Fréchet differentiable in  $H$  with  $D^2f(x) \in L_2(H)$ ,  $x \in H$ , and  $D^2f : H \rightarrow L_2(H)$  is bounded and continuous.

## 2 Preliminaries

Here we review basic facts on Lévy processes and introduce the Ornstein-Uhlenbeck process with values in  $\mathbb{R}^d$ . We refer to [Sa99], [Ap09] and [SY84] for more details.

We fix a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  which satisfies the usual assumptions (see, for instance, page 72 in [Ap09]). An  $(\mathcal{F}_t)$ -adapted  $d$ -dimensional stochastic process  $Z = (Z_t) = (Z_t)_{t \geq 0}$ ,  $d \geq 1$ , is a *Lévy process* if it is continuous in probability, it has stationary increments, càdlàg trajectories,  $Z_t - Z_s$  is independent of  $\mathcal{F}_s$ ,  $0 \leq s \leq t$ , and  $Z_0 = 0$ . Recall that there exists a unique  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$  such that

$$\mathbb{E}[e^{i\langle u, Z_t \rangle}] = e^{-t\psi(u)}, \quad u \in \mathbb{R}^d, \quad t \geq 0;$$

$\psi$  is called the *exponent* (or symbol) of  $Z$  ( $\mathbb{E}$  denotes expectation with respect to  $\mathbb{P}$ ). The *Lévy-Khintchine representation* for  $\psi$  is

$$\psi(u) = \frac{1}{2} \langle Qu, u \rangle - i \langle a, u \rangle - \int_{\mathbb{R}^d} \left( e^{i\langle u, y \rangle} - 1 - i \langle u, y \rangle \mathbf{1}_{\{|y| \leq 1\}}(y) \right) \nu(dy), \quad (2.1)$$

$u \in \mathbb{R}^d$ , where  $Q$  is a symmetric  $d \times d$  non-negative definite matrix,  $a \in \mathbb{R}^d$  (if  $B \subset \mathbb{R}^d$ ,  $\mathbf{1}_B(x) = 1$  if  $x \in B$  and  $\mathbf{1}_B(x) = 0$  if  $x \notin B$ ). Moreover,  $\nu$  is the *Lévy (jump) measure* (or intensity measure) of  $Z$ . Thus,  $\nu$  is a  $\sigma$ -finite (Borel) measure on  $\mathbb{R}^d$ , such that (1.2) holds. Note that

$$\mathbb{E}[|Z_t|] < \infty, \quad t \geq 0, \quad \text{if and only if} \quad \int_{\{|y| > 1\}} |y| \nu(dy) < \infty \quad (2.2)$$

(see [Ap09, Theorem 2.5.2]). The *Ornstein-Uhlenbeck process* which solves (1.3) is given by

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} dZ_s = e^{tA}x + Y_t, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (2.3)$$

where  $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$  and the stochastic convolution  $Y_t$  can be defined as a limit in probability of suitable Riemann sums (cf. page 104 in [Sa99] and [Ch87]). Let us denote by  $\mu_t$  the law of  $Y_t$ . The law  $\mu_t^x$  of  $X_t^x$  has characteristic function (or Fourier transform)  $\hat{\mu}_t^x$  given by

$$\hat{\mu}_t^x(h) = \mathbb{E}[e^{i\langle X_t^x, h \rangle}] = e^{i\langle e^{tA}x, h \rangle} \hat{\mu}_t(h) = e^{i\langle e^{tA^*}h, x \rangle} \exp\left(-\int_0^t \psi(e^{sA^*}h) ds\right), \quad (2.4)$$

$h \in \mathbb{R}^d$ , where  $A^*$  denotes the adjoint matrix of  $A$  and  $\psi$  is the exponent of  $Z$  (cf. [Ma04, Proposition 2.1]; clearly,  $\mu_t = \mu_t^0$ ). Next we recall basic facts about OU semigroups  $(P_t)$ . We define for  $f \in B_b(\mathbb{R}^d)$ ,  $x \in \mathbb{R}^d$ ,  $t \geq 0$

$$P_t f(x) = (P_t f)(x) = \mathbb{E}[f(X_t^x)] = \int_{\mathbb{R}^d} f(e^{tA}x + y) \mu_t(dy). \quad (2.5)$$

An important property is that, for any  $f \in C_b(\mathbb{R}^d)$ , the mapping:

$$(t, x) \mapsto P_t f(x) \text{ is continuous on } [0, +\infty) \times \mathbb{R}^d \text{ and} \quad (2.6)$$

$$\lim_{t \rightarrow 0^+} P_t f = f, \quad \text{uniformly on compact sets of } \mathbb{R}^d \quad (2.7)$$

(we refer to [BRS96, Lemma 2.1] which contains a more general result; see also [FR00, Section 4] and [Ap07, Theorem 4.1]). Note that (2.6) implies (2.7).

The spaces  $B_b(\mathbb{R}^d)$ ,  $C_b(\mathbb{R}^d)$ ,  $UC_b(\mathbb{R}^d)$  and  $C_0(\mathbb{R}^d)$  are all invariant for the OU semigroup (for instance,  $P_t(C_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$ ,  $t \geq 0$ ). Moreover,  $(P_t)$  is a  $C_0$ -semigroup of contractions on  $C_0(\mathbb{R}^d)$  (see [SY84], [Ma04] and [Tr12]), i.e.,

$$\lim_{t \rightarrow 0^+} \|P_t f - f\|_0 = 0, \quad f \in C_0(\mathbb{R}^d). \quad (2.8)$$

## 3 Ornstein-Uhlenbeck operators in $\mathbb{R}^d$

### 3.1 Classical solvability of the Cauchy problem

We show well-posedness of the Cauchy problem (1.5) involving  $\mathcal{L}_0$ . We write

$$\begin{aligned} \mathcal{L}_0 f(x) &= \langle Ax, Df(x) \rangle + \mathcal{L}_1 f(x), \quad \text{where} \\ \mathcal{L}_1 f(x) &= \int_{\mathbb{R}^d} (f(x+y) - f(x) - \mathbf{1}_{\{|y| \leq 1\}} \langle y, Df(x) \rangle) \nu(dy) \\ &\quad + \frac{1}{2} \text{Tr}(QD^2 f(x)) + \langle a, Df(x) \rangle, \quad f \in C_b^2(\mathbb{R}^d), \quad x \in \mathbb{R}^d. \end{aligned} \quad (3.1)$$

A *bounded classical solution*  $u$  to the Cauchy problem (1.5) is a bounded and continuous real function defined on  $E = [0, +\infty) \times \mathbb{R}^d$ , such that

- (i) there exist classical partial derivatives  $\partial_{x_i} u$  and  $\partial_{x_i x_j}^2 u$ ,  $i, j = 1, \dots, d$ , which are bounded and continuous on  $E$ ;
- (ii)  $u(\cdot, x)$  is a  $C^1$ -function on  $[0, +\infty)$ ,  $x \in \mathbb{R}^d$ , and  $u$  solves (1.5).

Existence of classical solutions is based on the following result.

**Theorem 3.1.** *Let  $f \in C_b^2(\mathbb{R}^d)$ . Then, for any  $x \in \mathbb{R}^d$ , the mapping:  $t \mapsto \mathcal{L}_0(P_t f)(x)$  is continuous on  $[0, +\infty)$  and  $\lim_{t \rightarrow 0^+} \mathcal{L}_0(P_t f)(x) = \mathcal{L}_0 f(x)$ . Moreover,*

$$P_t f(x) = f(x) + \int_0^t \mathcal{L}_0(P_s f)(x) ds, \quad t \geq 0, x \in \mathbb{R}^d. \quad (3.2)$$

In order to prove this result we introduce the linear span  $V(\mathbb{R}^d)$  of the real and imaginary parts of the functions  $x \mapsto e^{i\langle x, h \rangle}$ ,  $h \in \mathbb{R}^d$ .

We need an approximation result with functions in  $V(\mathbb{R}^d)$ . This is similar to [Da04, Proposition 2.67] and [Ma08, Proposition 4.2] (the main difference is that these results do not consider approximations of second derivatives). A detailed proof is given in [Tr12]. We give a sketch of proof in Appendix.

**Lemma 3.2.** *Let  $f \in C_b^2(\mathbb{R}^d)$ . There exist a double sequence  $(f_{nm})_{n,m \in \mathbb{N}} \subset V(\mathbb{R}^d)$  and a sequence  $(f_n)_{n \in \mathbb{N}} \subset C_b^2(\mathbb{R}^d)$  such that, for any  $n \geq 1$ ,*

$$\lim_{m \rightarrow \infty} \|f_{nm} - f_n\|_0 = 0, \quad \lim_{m \rightarrow \infty} \|Df_{nm} - Df_n\|_0 = 0, \quad \lim_{m \rightarrow \infty} \|D^2 f_{nm} - D^2 f_n\|_0 = 0.$$

Moreover, there exists  $M = M(f) > 0$  such that, for any  $n, m \geq 1$ ,

$$\|f_{nm}\|_0 + \|Df_{nm}\|_0 + \|D^2 f_{nm}\|_0 + \|f_n\|_0 + \|Df_n\|_0 + \|D^2 f_n\|_0 \leq M \quad (3.3)$$

and, for any  $x \in \mathbb{R}^d$ ,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad \lim_{n \rightarrow \infty} Df_n(x) = Df(x), \quad \lim_{n \rightarrow \infty} D^2 f_n(x) = D^2 f(x).$$

*Proof of Theorem 3.1.* The first assertion about the continuity of  $t \mapsto \mathcal{L}_0(P_t f)(x)$  follows easily using property (2.6) together with the following identity

$$\begin{aligned} \mathcal{L}_0(P_t f)(x) &= \langle e^{tA} Ax, P_t Df(x) \rangle + \\ &+ \int_{\mathbb{R}^d} (P_t f(x+y) - P_t f(x) - \mathbf{1}_{\{|y| \leq 1\}} \langle e^{tA} y, P_t Df(x) \rangle) \nu(dy) + \\ &+ \frac{1}{2} \text{Tr}(Q e^{tA*} P_t D^2 f(x) e^{tA}) + \langle e^{tA} a, P_t Df(x) \rangle \end{aligned} \quad (3.4)$$

(note that  $P_t Df(x) = \mathbb{E}[Df(X_t^x)]$  and  $P_t D^2 f(x) = \mathbb{E}[D^2 f(X_t^x)]$ ).

We will first prove (3.2) for  $f \in V(\mathbb{R}^d)$ . Then when  $f \in C_b^2(\mathbb{R}^d)$  we will use an approximating argument which is based on the previous lemma. We split the proof into two parts.

*I Step.* We show (3.2) for  $f \in V(\mathbb{R}^d)$ . It suffices to check that (3.2) holds when

$$f(x) = e^{i\langle h, x \rangle},$$

where  $h \in \mathbb{R}^d$ . When  $x \in \mathbb{R}^d$  is fixed, we have (see (2.4))

$$\begin{aligned} \left. \frac{\partial}{\partial t} (P_t f)(x) \right|_{t=0} &= \left. \frac{\partial}{\partial t} (\mathbb{E}[e^{i\langle h, e^{tA} x + \int_0^t e^{(t-s)A} dZ_s \rangle}]) \right|_{t=0} = \\ &= i\langle h, e^{tA} Ax \rangle e^{i\langle h, e^{tA} x \rangle} e^{-\int_0^t \psi(e^{sA*} h) ds} - e^{i\langle h, e^{tA} x \rangle} \psi(e^{tA*} h) e^{-\int_0^t \psi(e^{sA*} h) ds} \Big|_{t=0} = \\ &= \langle Df(x), Ax \rangle - e^{i\langle h, x \rangle} \psi(h), \end{aligned}$$



where  $\psi$  is the exponent of the Lévy process  $(Z_t)$ . Using the Lévy-Khintchine formula, we have that (cf. (3.1))

$$-e^{i\langle h, x \rangle} \psi(h) = \mathcal{L}_1(e^{i\langle h, \cdot \rangle})(x) = (\mathcal{L}_1 f)(x),$$

and therefore,

$$\left. \frac{\partial}{\partial t} (P_t f)(x) \right|_{t=0} = \mathcal{L}_0 f(x). \quad (3.5)$$

Similarly, recalling that  $Y_t = \int_0^t e^{(t-s)A} dZ_s$ , we can compute the derivative for  $t > 0$ :

$$\begin{aligned} \left. \frac{\partial}{\partial t} (P_t f)(x) \right|_{t>0} &= \left. \frac{\partial}{\partial t} (e^{i\langle h, e^{tA} x \rangle} \mathbb{E}[e^{i\langle h, Y_t \rangle}]) \right|_{t>0} = \\ &= i\langle h, e^{tA} A x \rangle e^{i\langle h, e^{tA} x \rangle} e^{\int_0^t \psi(e^{sA^*} h) ds} + e^{i\langle h, e^{tA} x \rangle} \psi(e^{tA^*} h) e^{\int_0^t \psi(e^{sA^*} h) ds}, \end{aligned} \quad (3.6)$$

using that  $e^{i\langle h, e^{tA} x \rangle} = e^{i\langle e^{tA^*} h, x \rangle}$ , we find that  $\frac{\partial}{\partial t} (P_t f)(x) = \mathcal{L}_0 (P_t f)(x)$ ,  $t \geq 0$ . Integrating with respect to  $t$ , we get the assertion.

*II Step.* We prove (3.2) when  $f \in C_b^2(\mathbb{R}^d)$ . We choose an approximating sequence  $(f_{nm})_{n,m \in \mathbb{N}} \subset V(\mathbb{R}^d)$  as in Lemma 3.2. We can write

$$P_t f_{nm}(x) = f_{nm}(x) + \int_0^t \mathcal{L}_0(P_s f_{nm})(x) ds, \quad t \geq 0, x \in \mathbb{R}^d, \quad (3.7)$$

for any  $n, m \in \mathbb{N}$ . In order to pass to the limit in (3.7) we fix  $T > 0$ ,  $x \in \mathbb{R}^d$ , and study the convergence of  $\mathcal{L}_0(P_s f_{nm})(x)$ , with  $s \in [0, T]$ . The term

$$\int_{\mathbb{R}^d} (P_s f_{nm}(x+y) - P_s f_{nm}(x) - \mathbf{1}_{\{|y| \leq 1\}} \langle y, DP_s f_{nm}(x) \rangle) \nu(dy)$$

can be written as  $I_{nm}(x) + J_{nm}(x)$ , where

$$I_{nm}(x) = \int_{\{|y| > 1\}} (P_s f_{nm}(x+y) - P_s f_{nm}(x)) \nu(dy) \quad \text{and} \quad (3.8)$$

$$J_{nm}(x) = \int_{\{|y| \leq 1\}} (P_s f_{nm}(x+y) - P_s f_{nm}(x) - \langle y, DP_s f_{nm}(x) \rangle) \nu(dy).$$

Passing to the limit first as  $m \rightarrow \infty$  and then as  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} I_{nm}(x)) = \int_{\{|y| > 1\}} (P_s f(x+y) - P_s f(x)) \nu(dy). \quad (3.9)$$

(recall that  $\nu$  is a finite measure on  $\{|y| > 1\}$ ). To deal with  $J_{nm}(x)$  we first observe that, using Taylor formula,

$$\begin{aligned} |P_s f_{nm}(x+y) - P_s f_{nm}(x) - \langle y, DP_s f_{nm}(x) \rangle| &\leq |y|^2 \|D^2 P_s f_{nm}\|_0 \leq \\ &\leq C_T |y|^2 \|D^2 f_{nm}\|_0 \leq C_T M |y|^2, \quad y \in \mathbb{R}^d. \end{aligned}$$



Moreover, by Lemma 3.2, as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ , we have that

$$\langle y, DP_s f_{nm}(x) \rangle = \langle e^{sA} y, P_s D f_{nm}(x) \rangle \rightarrow \langle e^{sA} y, P_s D f(x) \rangle = \langle y, DP_s f(x) \rangle, \quad (3.10)$$

$y \in \mathbb{R}^d$ . Therefore, as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ , we find

$$\lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} J_{nm}(x)) = \int_{\{|y| \leq 1\}} (P_s f(x+y) - P_s f(x) - \langle y, DP_s f(x) \rangle) \nu(dy).$$

Similarly, for the other terms of  $\mathcal{L}_0(P_s f_{nm})$  we have that

$$\begin{aligned} \frac{1}{2} \text{Tr}(QD^2 P_s f_{nm}(x)) &= \frac{1}{2} \text{Tr}(Qe^{sA*} P_s D^2 f_{nm}(x) e^{sA}) \rightarrow \\ &\rightarrow \frac{1}{2} \text{Tr}(Qe^{sA*} P_s D^2 f(x) e^{sA}) = \frac{1}{2} \text{Tr}(QD^2 P_s f(x)), \end{aligned} \quad (3.11)$$

as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ , and

$$\langle a, DP_s f_{nm}(x) \rangle = \langle e^{sA} a, P_s D f_{nm}(x) \rangle \rightarrow \langle e^{sA} a, P_s D f(x) \rangle = \langle a, DP_s f(x) \rangle, \quad (3.12)$$

as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ , for  $s \in [0, T]$ . We also observe that  $\|\mathcal{L}_1(P_s f_{nm})\|_0 \leq M_T$ , for  $s \in [0, T]$ . It follows that

$$\int_0^t \mathcal{L}_1(P_s f_{nm})(x) ds \rightarrow \int_0^t \mathcal{L}_1(P_s f)(x) ds, \quad t \in [0, T],$$

as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ . Finally, for any  $x \in \mathbb{R}^d$  and  $s \in [0, T]$ ,

$$\langle Ax, DP_s f_{nm}(x) \rangle = \langle e^{sA} Ax, P_s (D f_{nm})(x) \rangle \rightarrow \langle e^{sA} Ax, P_s (D f)(x) \rangle = \langle Ax, DP_s f(x) \rangle. \quad (3.13)$$

Therefore, for  $t \in [0, T]$ , as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ ,  $\int_0^t \langle Ax, DP_s f_{nm}(x) \rangle ds \rightarrow \int_0^t \langle Ax, DP_s f(x) \rangle ds$ . In conclusion, passing to the limit as  $m \rightarrow \infty$  and  $n \rightarrow \infty$  in (3.7), we obtain (3.2) and the proof is complete.  $\square$

**Theorem 3.3.** *Let  $f \in C_b^2(\mathbb{R}^d)$ . If we set  $u(t, x) = P_t f(x)$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^d$ , where  $(P_t)$  is given in (2.5), then  $u$  is the unique bounded classical solution to the Cauchy problem (1.5).*

*Proof. Existence.* By (2.6) we know that  $u(t, x) = P_t f(x)$  is bounded and continuous on  $[0, +\infty) \times \mathbb{R}^d$ . Moreover, differentiating under the integral sign it is easy to see that there exist classical partial derivatives  $\partial_{x_i} u$  and  $\partial_{x_i x_j}^2 u$  which are bounded and continuous on  $[0, \infty) \times \mathbb{R}^d$ .

We have also to verify that  $u(\cdot, x)$  is a  $C^1$ -function on  $[0, +\infty)$ , for any  $x \in \mathbb{R}^d$ . This follows easily from (3.2) and (3.4) using again (2.6). Theorem 3.1 also shows that  $u$  solves (1.5).

*Uniqueness.* Let  $u$  be a bounded classical solution to (1.5). To prove uniqueness we will use a quite standard probabilistic argument based on the Itô formula (cf. [Ap09, Section 4.4]). First we recall the Lévy-Itô decomposition formula (see [Sa99, Chapter 4] or [Ap09, Chapter 2]). According to (2.1) this formula says that on the fixed stochastic basis there exist a  $Q$ -Wiener process  $W^Q = (W_t^Q)$

with covariance matrix  $Q$  and an independent Poisson random measure  $N$  on  $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$  with intensity measure  $l \otimes \nu$  (here  $l$  is Lebesgue measure on  $\mathbb{R}_+$ ) such that

$$Z_t = at + W_t^Q + \int_0^t \int_{\{|x| \leq 1\}} x \tilde{N}(ds, dx) + \int_0^t \int_{\{|x| > 1\}} z N(ds, dx), \quad (3.14)$$

$t \geq 0$ ; here  $\tilde{N}$  is the compensated Poisson measure (i.e.,  $\tilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx)$ ). We apply the Itô formula to the Ornstein-Uhlenbeck process  $(X_t^x)$ . We fix  $t > 0$  and define, for  $s \in [0, t]$ ,  $x \in \mathbb{R}^d$ ,  $v(s, x) = u(t - s, x)$ . We have

$$\begin{aligned} v(t, X_t^x) - v(0, x) &= f(X_t^x) - u(t, x) \\ &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(t - r, X_{r-}^x + x) - u(t - r, X_{r-}^x)] \tilde{N}(dr, dx) \\ &\quad + \int_0^t \langle Du(t - r, X_r^x), dW_r^Q \rangle + \int_0^t (-\partial_s u(t - r, X_r^x) + \mathcal{L}_0 u(t - r, X_r^x)) dr. \end{aligned}$$

Since the last integral is zero, by taking the expectation, we get  $\mathbb{E}[f(X_t^x)] = u(t, x)$ , and, therefore,  $u(t, x) = P_t f(x)$ ,  $x \in \mathbb{R}^d$ .  $\square$

### 3.2 An invariant core in $C_0(\mathbb{R}^d)$

In this section we study the Ornstein-Uhlenbeck semigroup  $(P_t)$  acting on  $C_0(\mathbb{R}^d)$ . This is a strongly continuous semigroup or a  $C_0$ -semigroup of contractions (cf. (2.8)). We determine a core  $\mathcal{D}_0$  which is invariant for the semigroup.

Let us denote by  $\mathcal{L}$  the generator of  $(P_t)$ , i.e.,  $\mathcal{L}f = \lim_{t \rightarrow 0} \frac{P_t f - f}{t}$ , with domain  $D(\mathcal{L})$  being the set of all  $f \in C_0(\mathbb{R}^d)$  such that the previous limit exists as a limit in  $C_0(\mathbb{R}^d)$  (see, for instance, [EN99] for the theory of linear  $C_0$ -semigroups). Let us recall the general definition of core for a  $C_0$ -semigroup (cf. [Sa99, Section 31]).

Let  $(S_t)$  be a  $C_0$ -semigroup on a Banach space  $X$  with generator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \rightarrow X$ . A core for  $(S_t)$  (or for  $\mathcal{A}$ ) is a subspace  $\mathcal{D} \subset D(\mathcal{A})$  such that for every  $\psi \in D(\mathcal{A})$  there exists a sequence  $(\psi_n) \subset \mathcal{D}$  which verifies:  $\psi_n \rightarrow \psi$  and  $\mathcal{A}\psi_n \rightarrow \mathcal{A}\psi$  in  $X$ .

Using also Theorem 3.1 we show that the space

$$\mathcal{D}_0 = \{f \in C_0^2(\mathbb{R}^d) : x \mapsto \langle Ax, Df(x) \rangle \in C_0(\mathbb{R}^d)\}$$

is an invariant core for  $(P_t)$ .

**Theorem 3.4.** *Let  $\mathcal{L}$  be the generator of the OU semigroup  $(P_t)$  in  $C_0(\mathbb{R}^d)$ . The following statements hold:*

- (i)  $\mathcal{D}_0 \subset D(\mathcal{L})$  and  $\mathcal{L}f = \mathcal{L}_0 f$ , for any  $f \in \mathcal{D}_0$  ( $\mathcal{L}_0$  is defined in (1.1));
- (ii)  $\mathcal{D}_0$  is invariant for  $(P_t)$ , i.e.,  $P_t(\mathcal{D}_0) \subset \mathcal{D}_0$ ,  $t \geq 0$ ;
- (iii)  $\mathcal{D}_0$  is a core for  $\mathcal{L}$ .

*Proof.* (i) Let  $f \in \mathcal{D}_0$ . We will use Itô's formula (see [Ap09, Section 4.4]). Since in particular  $f \in C_b^2(\mathbb{R}^d)$  we have  $\mathbb{P}$ -a.s. (cf. (3.14))

$$\begin{aligned} f(X_t^x) &= f(x) + \int_0^t \mathcal{L}_0 f(X_s^x) ds \\ &+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f(X_{s-}^x + y) - f(X_{s-}^x)] \tilde{N}(ds, dy) + \int_0^t \langle Df(X_s^x), dW_s^Q \rangle, \quad x \in \mathbb{R}^d, \end{aligned} \quad (3.15)$$

$t \geq 0$ . Since  $\mathcal{L}_0 f$  is a bounded function by taking the expectation we get (recall that  $f$  and  $Df$  are bounded and so the stochastic integrals have both mean zero)

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{L}_0 f(x) ds, \quad x \in \mathbb{R}^d. \quad (3.16)$$

We can write for  $t > 0$ ,  $x \in \mathbb{R}^d$ ,

$$\frac{P_t f(x) - f(x)}{t} - \mathcal{L}_0 f(x) = \frac{1}{t} \int_0^t [P_s \mathcal{L}_0 f(x) - \mathcal{L}_0 f(x)] ds. \quad (3.17)$$

From this formula it is easy to deduce that  $f \in D(\mathcal{L})$  and also that  $\mathcal{L}f = \mathcal{L}_0 f$ .

(ii) Differentiating under the integral sign one checks that  $P_t(C_0^2(\mathbb{R}^d)) \subset C_0^2(\mathbb{R}^d)$ ,  $t \geq 0$ . Thus to prove that  $\mathcal{D}_0$  is invariant for the semigroup, it is enough to show that for  $f \in \mathcal{D}_0$ ,  $t \geq 0$ , we have that

$$x \mapsto \langle Ax, DP_t f(x) \rangle \in C_0(\mathbb{R}^d). \quad (3.18)$$

To check (3.18) we use Theorem 3.1 and (3.16). We know that for  $x \in \mathbb{R}^d$ ,  $t \geq 0$ ,

$$\int_0^t P_s \mathcal{L}_0 f(x) ds = \int_0^t \mathcal{L}_0 P_s f(x) ds.$$

Since  $s \mapsto \mathcal{L}_0 P_s f(x)$  and  $s \mapsto P_s \mathcal{L}_0 f(x)$  are both continuous functions we get

$$\mathcal{L}_0 P_t f(x) = P_t \mathcal{L}_0 f(x), \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

Let us fix  $t > 0$ . The previous identity shows in particular that  $\mathcal{L}_0 P_t f \in C_0(\mathbb{R}^d)$ . We have (see (3.1))

$$\mathcal{L}_0 P_t f(x) = \mathcal{L}_1 P_t f(x) + \langle Ax, DP_t f(x) \rangle, \quad x \in \mathbb{R}^d,$$

and since  $P_t f \in C_0(\mathbb{R}^d)$  one can easily check that  $\mathcal{L}_1 P_t f \in C_0(\mathbb{R}^d)$  (cf. the proof of [Sa99, Theorem 31.5]). It follows that  $\langle A(\cdot), DP_t f(\cdot) \rangle \in C_0(\mathbb{R}^d)$  and this gives (3.18).

(iii) We can use a well-known criterium for the existence of a core (see [EN99, Proposition II.1.7]). First  $\mathcal{D}_0$  is dense in  $C_0(\mathbb{R}^d)$  (to this purpose note that  $C_K^2(\mathbb{R}^d)$  is contained in  $\mathcal{D}_0$ ); then  $\mathcal{D}_0$  is invariant for the semigroup by (ii). It follows that  $\mathcal{D}_0$  is a core for  $\mathcal{L}$ . This completes the proof.  $\square$

**Corollary 3.5.** *If  $f \in \mathcal{D}_0$ , then*

$$P_t f(x) = f(x) + \int_0^t P_s (\mathcal{L}_0 f)(x) ds = f(x) + \int_0^t \mathcal{L}_0 (P_s f)(x) ds, \quad t \geq 0, x \in \mathbb{R}^d.$$

*Proof.* From a general result of semigroup theory the previous formula holds when  $\mathcal{L}_0$  is replaced by the generator  $\mathcal{L}$  and  $f \in D(\mathcal{L})$ . Using Theorem 3.4 we easily obtain the assertion.  $\square$

**Remark 3.6.** We have mentioned formula (1.7) proved in [SY84, Theorem 3.1]. This implies that  $C_K^2(\mathbb{R}^d) \subset D(\mathcal{L})$  and  $\mathcal{L}_0 f = \mathcal{L}f$ ,  $f \in C_K^2(\mathbb{R}^d)$ . However Theorem 3.3 can not be deduced by [SY84, Theorem 3.1] even if we require that  $f$  belongs to  $C_K^2(\mathbb{R}^d)$ . This is because the space  $C_K^2(\mathbb{R}^d)$  is not invariant for  $(P_t)$ . From formula (1.7) we only get

$$P_t f(x) = f(x) + \int_0^t \mathcal{L}(P_s f)(x) ds, \quad t \geq 0, x \in \mathbb{R}^d, f \in C_K^2(\mathbb{R}^d).$$

## 4 Infinite dimensional Ornstein-Uhlenbeck processes

We consider a real and separable Hilbert space  $H$  with norm  $|\cdot|$  and inner product  $\langle \cdot, \cdot \rangle$ . We fix a Lévy process  $Z = (Z_t)$  with values in  $H$  (see [PZ07, Chapter 4]). Similarly to the case when  $H = \mathbb{R}^d$ ,  $Z$  is an  $H$ -valued process defined on some stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , continuous in probability, having stationary independent increments, càdlàg trajectories, and such that  $Z_0 = 0$ . One has that

$$\mathbb{E}[e^{i\langle Z_t, h \rangle}] = \exp(-t\psi(h)), \quad h \in H, t \geq 0,$$

where the exponent  $\psi : H \rightarrow \mathbb{C}$  is defined similarly to (2.1) as

$$\psi(h) = \frac{1}{2} \langle Qh, h \rangle - i \langle a, h \rangle - \int_H \left( e^{i\langle h, y \rangle} - 1 - i \langle h, y \rangle \mathbf{1}_{\{|y| \leq 1\}}(y) \right) \nu(dy); \quad (4.1)$$

here  $Q : H \rightarrow H$  is a non-negative symmetric *trace-class* operator,  $a \in H$ , and  $\nu$  is the Lévy (jump) measure of  $Z$  (i.e.,  $\nu$  is a  $\sigma$ -finite (Borel) measure on  $H$ , such that (1.2) holds with  $\mathbb{R}^d$  replaced by  $H$ ).

Note that a Lévy-Itô decomposition formula as (3.14) holds also in infinite dimensions:

$$Z_t = at + W_t^Q + \int_0^t \int_{\{|x| \leq 1\}} x \tilde{N}(ds, dx) + \int_0^t \int_{\{|x| > 1\}} x N(ds, dx), \quad (4.2)$$

$t \geq 0$ , where  $N$  is the Poisson random measure associated to  $Z$ ,  $a \in H$  and  $W^Q = (W_t^Q)$  is a  $Q$ -Wiener process with values in  $H$  which is independent of  $N$  (cf. Section 2 in [Ap07]).

Let  $A : D(A) \subset H \rightarrow H$  be the generator of a  $C_0$ -semigroup  $(e^{tA}) = (e^{tA})_{t \geq 0}$  on  $H$ . By  $A^* : D(A^*) \subset H \rightarrow H$  we denote its adjoint operator which generates the  $C_0$ -semigroup  $(e^{tA^*})$ .

We will deal with the following generalization of the *Ornstein-Uhlenbeck process* considered in (2.3):

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} dZ_s = e^{tA}x + Y_t, \quad t \geq 0, x \in H. \quad (4.3)$$

The stochastic integral is still a limit in probability of suitable Riemann sums (we refer to [Ch87] and [PZ07]). The associated *Ornstein-Uhlenbeck semigroup*  $(P_t)$  is still defined as

$$P_t f(x) = \mathbb{E}[f(X_t^x)] = \int_H f(e^{tA}x + y)\mu_t(dy), \quad t \geq 0, \quad x \in H, \quad f \in B_b(H). \quad (4.4)$$

where  $\mu_t$  is the law of  $Y_t$  and has characteristic function

$$\hat{\mu}_t(h) = \mathbb{E}[e^{i\langle Y_t, h \rangle}] = \exp\left(-\int_0^t \psi(e^{sA^*}h)ds\right), \quad h \in H, \quad t \geq 0 \quad (4.5)$$

(cf. [Ch87, Corollary 1.7] or [FR00]). In contrast with Section 3.2 if  $H$  is infinite dimensional then the space  $C_0(H)$  is in general not invariant for the OU semigroup (see page 91 in [Ap07] for more details). Hence it is convenient to deal with  $(P_t)$  acting on  $C_b(H)$  or  $UC_b(H)$  since both spaces are invariant (however recall that  $(P_t)$  is not strongly continuous neither on  $C_b(H)$  nor on  $UC_b(H)$  if we consider the sup-norm topology; see [Ce94]).

It is important to note that (2.6) and (2.7) holds for  $(P_t)$  even in this infinite dimensional setting, i.e., for any  $f \in C_b(H)$ , the real mapping:

$$(t, x) \mapsto P_t f(x) \text{ is continuous on } [0, +\infty) \times \mathbb{R}^d \text{ and} \quad (4.6)$$

$$\lim_{t \rightarrow 0^+} P_t f = f, \quad \text{uniformly on compact sets of } H. \quad (4.7)$$

We refer to [BRS96, Lemma 2.1]. To this purpose note that  $t \mapsto \mu_t$  is continuous with respect to the weak topology of Borel probability measures on  $H$  by [Pa67, Lemma VI.2.1]. Indeed for any  $h \in H$ ,  $t \mapsto \hat{\mu}_t(h)$  is continuous on  $[0, +\infty)$  and moreover, according to page 19 in [FR00], for any  $T > 0$ , the family  $(\mu_t)_{t \in [0, T]}$  is tight. Results similar to (4.6) and (4.7) are proved in [FR00, Theorem 4.2] and in [Ap07, Theorem 4.1].

To study the OU semigroup it is useful to fix a notion of pointwise convergence of functions (see [EK86], [Pr99], [Da04] and [Ma08]).

A sequence  $(f_n) \subset C_b(H)$  is said to be  *$\pi$ -convergent* to a map  $f \in C_b(H)$  and we shall write

$$f_n \xrightarrow{\pi} f \quad (4.8)$$

as  $n \rightarrow \infty$  (or  $\lim_{n \rightarrow \infty} f_n \stackrel{\pi}{=} f$ ) if it converges boundedly and pointwise, i.e.,  $\sup_{n \geq 1} \sup_{x \in H} |f_n(x)| = \sup_{n \geq 1} \|f_n\|_0 < \infty$  and  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ ,  $x \in H$ .

We mention that a related notion of uniform convergence on compact sets can also be used (cf. [Ce94] and [CG95]).

## 5 The Cauchy problem in infinite dimensions

Let us introduce the infinite-dimensional OU operator  $\mathcal{L}_0$  associated to the OU process introduced in Section 4:

$$\begin{aligned} \mathcal{L}_0 f(x) &= \langle Ax, Df(x) \rangle + \mathcal{L}_1 f(x); \\ \mathcal{L}_1 f(x) &= \int_H (f(x+y) - f(x) - \mathbf{1}_{\{|y| \leq 1\}} \langle y, Df(x) \rangle) \nu(dy) \\ &\quad + \frac{1}{2} \text{Tr}(QD^2 f(x)) + \langle a, Df(x) \rangle, \end{aligned} \quad (5.1)$$

where  $f \in C_b^2(H)$ ,  $x \in D(A)$  ( $a$ ,  $Q$  and  $\nu$  are given in the Lévy-Khintchine formula (4.1)). Note that by a well-known result  $QD^2f(x)$  is a trace class operator, for any  $x \in H$  (see, for instance, [DZ92, Appendix C]). We consider an infinite dimensional Cauchy problem which generalizes (1.5), i.e.,

$$\begin{cases} \partial_t u(t, x) = \mathcal{L}_0 u(t, x) \\ u(0, x) = f(x). \end{cases} \quad (5.2)$$

Let us introduce the space

$$C_A^2(H) = \{f \in C_b^2(H) : Df(x) \in D(A^*), x \in H, \text{ and } A^*Df \in C_b(H, H)\}. \quad (5.3)$$

A similar space has been considered in [GK01] and [Ap09]. However in contrast with to [GK01] and [Ap09] we do not require that the mapping  $x \mapsto \langle x, A^*Df(x) \rangle$  belongs to  $C_b(H)$ . Clearly, when  $H = \mathbb{R}^d$  the space  $C_A^2(H)$  coincides with  $C_b^2(\mathbb{R}^d)$ . Note that if  $f \in C_A^2(H)$  then

$$\mathcal{L}_0 f(x) = \langle x, A^*Df(x) \rangle + \mathcal{L}_1 f(x), \quad x \in H. \quad (5.4)$$

A *bounded classical solution*  $u$  to the Cauchy problem (5.2) is a bounded and continuous real function defined on  $E = [0, +\infty) \times H$ , such that

- (i)  $u(t, \cdot) \in C_A^2(H)$ ,  $t \geq 0$ , and  $D_x u$ ,  $A^*D_x u : E \rightarrow H$ ,  $D_x^2 u : E \rightarrow L_2(H)$  are bounded and continuous functions;
- (ii)  $u(\cdot, x)$  is a  $C^1$ -function on  $[0, +\infty)$ ,  $x \in H$ , and  $u$  solves (5.2).

To show solvability of (5.2) we first extend Theorem 3.1 to infinite dimensions.

**Theorem 5.1.** *Let  $f \in C_b^2(H)$ . Let  $(P_t)$  be the OU semigroup defined in (4.4). The following statements hold:*

- (i) *for any  $x \in D(A)$ , the real mapping  $t \mapsto \mathcal{L}_0(P_t f)(x)$  is continuous on  $[0, +\infty)$  and  $\lim_{t \rightarrow 0^+} \mathcal{L}_0(P_t f)(x) = \mathcal{L}_0 f(x)$ . Moreover, we have:*

$$P_t f(x) = f(x) + \int_0^t \mathcal{L}_0(P_s f)(x) ds, \quad t \geq 0, \quad x \in D(A); \quad (5.5)$$

- (ii) *if in addition  $f \in C_A^2(H)$ , then (i) and (5.5) hold for any  $x \in H$ .*

To prove the theorem we will use results from Section 3.1.

Recall that a function  $g : H \rightarrow \mathbb{R}$  is called *cylindrical* if there exist  $h_1, \dots, h_n \in H$  and  $l : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$g(x) = l(\langle x, h_1 \rangle, \dots, \langle x, h_n \rangle), \quad x \in H.$$

Let us fix an orthonormal basis  $(e_k)$  in  $H$  and consider the orthogonal projections  $P_n : H \rightarrow H$ ,

$$P_n x = \sum_{k=1}^n \langle x, e_k \rangle e_k, \quad x \in H. \quad (5.6)$$

We have the following quite standard approximation result.

**Lemma 5.2.** *Let  $f \in C_b^2(H)$  and consider the cylindrical functions  $(f_n) \subset C_b^2(H)$ ,  $f_n(x) = f(P_n x)$ ,  $x \in H$ ,  $n \geq 1$ . We have, for any  $x, h \in H$ , passing to the limit as  $n \rightarrow \infty$ ,*

$$f_n(x) \rightarrow f(x), \quad \langle Df_n(x), h \rangle \rightarrow \langle Df(x), h \rangle, \quad Df_n^2(x) \rightarrow D^2f(x) \text{ in } L_2(H);$$

$$\text{with } \sup_{n \geq 1} \sup_{y \in H} (|f_n(y)| + |Df_n(y)| + \|D^2f_n(y)\|_{HS}) = M < \infty.$$

*Proof.* Since  $P_n x \rightarrow x$  as  $n \rightarrow \infty$  and  $|P_n x| \leq |x|$ ,  $n \geq 1$ ,  $x \in H$ , it is easy to prove the assertions about  $f_n$  and  $Df_n$ . Let us only consider  $D^2f_n$  and fix  $x \in H$ . We have  $D^2f_n(x) = P_n D^2f(P_n x) P_n$  and

$$\begin{aligned} & \|P_n D^2f(P_n x) P_n - D^2f(x)\|_{HS}^2 \\ & \leq 2\|P_n[D^2f(P_n x) - D^2f(x)]P_n\|_{HS}^2 + 2\|P_n D^2f(x) P_n - D^2f(x)\|_{HS}^2 \\ & \leq 2\|D^2f(P_n x) - D^2f(x)\|_{HS}^2 + 2 \sum_{j,k > n} \langle D^2f(x) e_j, e_k \rangle^2. \end{aligned}$$

Using also the continuity of  $D^2f : H \rightarrow L_2(H)$  we find that

$$\|P_n D^2f(P_n x) P_n - D^2f(x)\|_{HS}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

To prove the uniform estimate on  $\|D^2f_n\|_0$  we note that  $\|P_n D^2f(P_n x) P_n\|_{HS} \leq \sup_{y \in H} \|D^2f(y)\|_{HS}$ , for any  $x \in H$ ,  $n \geq 1$ .  $\square$

*Proof of Theorem 5.1.* We follow the method of the proof of Theorem 3.1. We only indicate some changes.

First the assertion about the continuity of  $t \mapsto \mathcal{L}_0(P_t f)(x)$  can be proved by an identity like (3.4) with  $x \in D(A)$  and  $\mathbb{R}^d$  replaced by  $H$ .

(i) We split the proof of (i) into three parts.

*I Step.* We prove the assertion (5.5) when  $f$  is a cylindrical function of the form

$$f(x) = l(\langle x, e_1 \rangle, \dots, \langle x, e_n \rangle), \quad x \in H, \quad l \in V(\mathbb{R}^n),$$

where  $n \geq 1$  and  $(e_k)$  is an orthonormal basis in  $H$ . It is enough to consider  $l(x_1, \dots, x_n) = e^{i(h_1 x_1 + \dots + h_n x_n)}$ , for a fixed  $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ . Let us define  $Sh = \sum_{j=1}^n h_j e_j \in H$ . Note that

$$f(y) = e^{i\langle y, Sh \rangle}, \quad y \in H$$

(we are using the inner product in  $H$ ). Let  $x \in D(A)$ ; we have

$$\begin{aligned} \left. \frac{\partial}{\partial t} (P_t f)(x) \right|_{t=0} &= \left. \frac{\partial}{\partial t} (\mathbb{E}[e^{i\langle Sh, e^{tA} x + \int_0^t e^{(t-s)A} dZ_s \rangle}]) \right|_{t=0} = \\ &= i\langle Sh, e^{tA} Ax \rangle e^{i\langle Sh, e^{tA} x \rangle} e^{-\int_0^t \psi(e^{sA*} Sh) ds} - e^{i\langle Sh, e^{tA} x \rangle} \psi(e^{tA*} Sh) e^{-\int_0^t \psi(e^{sA*} Sh) ds} \Big|_{t=0} = \\ &= \langle Df(x), Ax \rangle - e^{i\langle Sh, x \rangle} \psi(Sh), \end{aligned}$$



where  $\psi$  is the exponent of the Lévy process  $(Z_t)_{t \geq 0}$  (see (4.1)). Using the Lévy-Khintchine formula, we find that

$$\left. \frac{\partial}{\partial t} (P_t f)(x) \right|_{t=0} = \mathcal{L}_0 f(x), \quad x \in D(A). \quad (5.7)$$

Similarly, recalling that  $Y_t = \int_0^t e^{(t-s)A} dZ_s$ , we can compute the derivative for  $t > 0$ :  $\frac{\partial}{\partial t} (P_t f)(x) = \mathcal{L}_0 (P_t f)(x)$ ,  $t \geq 0$ . Integrating with respect to  $t$ , we prove (5.5) for our cylindrical function  $f$ .

*II Step.* We prove the assertion when  $f : H \rightarrow \mathbb{R}$  is cylindrical of the form

$$f(x) = g(\langle x, e_1 \rangle, \dots, \langle x, e_N \rangle), \quad x \in H, \quad g \in C_b^2(\mathbb{R}^N),$$

for some  $N \geq 1$ . If  $R_N : H \rightarrow \mathbb{R}^N$ ,  $R_N x = (\langle x, e_1 \rangle, \dots, \langle x, e_N \rangle)$  then  $f(x) = g(R_N x)$ ,  $x \in H$ .

According to Lemma 3.2 we can choose  $(f_{nm})_{n,m \in \mathbb{N}} \subset V(\mathbb{R}^N)$  to approximate  $g$ . Let  $F_{nm} : H \rightarrow \mathbb{R}$ ,  $F_{nm}(x) = f_{nm}(R_N x)$  and similarly  $F_n(x) = f_n(R_N x)$ ,  $x \in H$ . By the previous step we can write

$$P_t F_{nm}(x) = F_{nm}(x) + \int_0^t \mathcal{L}_0(P_s F_{nm})(x) ds, \quad t \geq 0, x \in D(A), \quad (5.8)$$

for any  $n, m \in \mathbb{N}$ . Note that, for any  $n \geq 1$ ,

$$\begin{aligned} F_{nm} &\xrightarrow{\pi} F_n, & \langle DF_{nm}(\cdot), h \rangle &\xrightarrow{\pi} \langle DF_n(\cdot), h \rangle, \\ \langle D^2 F_{nm}(\cdot) h, k \rangle &\xrightarrow{\pi} \langle D^2 F_n(\cdot) h, k \rangle & \text{as } m \rightarrow \infty; \\ F_n &\xrightarrow{\pi} F, & \langle DF_n(\cdot), h \rangle &\xrightarrow{\pi} \langle DF(\cdot), h \rangle, \\ \langle D^2 F_n(\cdot) h, k \rangle &\xrightarrow{\pi} \langle D^2 F(\cdot) h, k \rangle & \text{as } n \rightarrow \infty, \end{aligned} \quad (5.9)$$

$h, k \in H$ , and

$$\sup_{y \in H} (|F_{nm}(y)| + |F_n(y)| + |DF_{nm}(y)| + |DF_n(y)| + \|D^2 F_{nm}(y)\|_L + \|D^2 F_n(y)\|_L) = C$$

where  $C$  is a constant independent of  $n$  and  $m$ . Thanks to the previous formulas in order to pass to the limit in (5.8) we fix  $T > 0$ ,  $x \in D(A)$ , and study the convergence of  $\mathcal{L}_0(P_s F_{nm})(x)$ , with  $s \in [0, T]$ . The term

$$\int_H (P_s F_{nm}(x+y) - P_s F_{nm}(x) - \mathbb{1}_{\{|y| \leq 1\}} \langle y, DP_s F_{nm}(x) \rangle) \nu(dy)$$

can be splitted as in (3.8); passing to the limit first as  $m \rightarrow \infty$  and then as  $n \rightarrow \infty$ , we get

$$\int_H (P_s F(x+y) - P_s F(x) - \mathbb{1}_{\{|y| \leq 1\}} \langle y, DP_s F(x) \rangle) \nu(dy).$$

Similarly, it follows that  $\int_0^t \mathcal{L}_1(P_s F_{nm})(x) ds \rightarrow \int_0^t \mathcal{L}_1(P_s F)(x) ds$ ,  $t \in [0, T]$ , first as  $m \rightarrow \infty$  and then as  $n \rightarrow \infty$ . Finally as in (3.13) we find, for  $s \in [0, T]$ ,

$$\langle Ax, DP_s F_{nm}(x) \rangle = \langle e^{sA} Ax, P_s(DF_{nm})(x) \rangle \rightarrow \langle Ax, DP_s F(x) \rangle,$$

first as  $m \rightarrow \infty$  and then as  $n \rightarrow \infty$ .

Moreover,  $|\langle Ax, DP_s F_{nm}(x) \rangle| + |\langle Ax, DP_s F_n(x) \rangle| \leq c_T |Ax|$ ,  $n, m \geq 1$ . Therefore, for  $t \in [0, T]$ , as  $m \rightarrow \infty$  and  $n \rightarrow \infty$ ,

$$\int_0^t \langle Ax, DP_s F_{nm}(x) \rangle ds \rightarrow \int_0^t \langle Ax, DP_s F(x) \rangle ds.$$

In conclusion, as  $m \rightarrow \infty$  and  $n \rightarrow \infty$  in (5.8), we obtain the assertion.

*III Step.* We prove (5.5) when  $f \in C_b^2(H)$ .

We consider a sequence of cylindrical functions  $(f_n)$  which approximate  $f$  as in Lemma 5.2. By the previous step we know that (5.5) holds when  $f$  is replaced by  $f_n$ ,  $n \geq 1$ , i.e.,

$$P_t f_n(x) = f_n(x) + \int_0^t \mathcal{L}_0(P_s f_n)(x) ds, \quad t \geq 0, \quad x \in D(A).$$

In order to pass to the limit as  $n \rightarrow \infty$  we proceed as in the previous step. The only difficulty concerns the term  $\frac{1}{2} \text{Tr}(QD^2 P_s f_n(x))$ . We have to justify the following limit

$$\begin{aligned} \frac{1}{2} \text{Tr}(QD^2 P_s f_n(x)) &= \frac{1}{2} \text{Tr}(Qe^{sA^*} P_s D^2 f_n(x) e^{sA}) \rightarrow \\ &\rightarrow \frac{1}{2} \text{Tr}(Qe^{sA^*} P_s D^2 f(x) e^{sA}) = \frac{1}{2} \text{Tr}(QD^2 P_s f(x)), \end{aligned} \quad (5.10)$$

as  $n \rightarrow \infty$ . To this purpose we use basic properties of trace class operators (cf. Appendix C in [DZ92])

$$\begin{aligned} |\text{Tr}(Qe^{sA^*} P_s [D^2 f_n(x) - D^2 f(x)] e^{sA})| &= |\text{Tr}(e^{sA} Qe^{sA^*} P_s [D^2 f_n(x) - D^2 f(x)])| \\ &\leq \|e^{sA} Qe^{sA^*}\|_{HS} \|D^2 f_n(x) - D^2 f(x)\|_{HS} \end{aligned}$$

which tends to 0 as  $n \rightarrow \infty$ . Note that we also have the bound

$$|\text{Tr}(QD^2 P_s f_n(x))| \leq |\text{Tr}(Qe^{sA^*} P_s D^2 f_n(x) e^{sA})| \leq C_T \sup_{y \in H} \|D^2 f_n(y)\|_{HS} \leq M_T,$$

for any  $t \in [0, T]$ ,  $n \geq 1$ . The assertion (i) is proved.

**(ii)** Let us fix  $x_0 \in H$ . There exists a sequence  $(x_n) \subset D(A)$  such that  $x_n \rightarrow x_0$ .

According to (i) and (5.1), we have, for any  $n \geq 1$ ,  $t \geq 0$ ,

$$\begin{aligned} P_t f(x_n) - f(x_n) &= \int_0^t \mathcal{L}_1(P_s f)(x_n) ds + \int_0^t \langle Ax_n, DP_s f(x_n) \rangle ds \\ &= \int_0^t \mathcal{L}_1(P_s f)(x_n) ds + \int_0^t \langle e^{sA} x_n, P_s (A^* Df)(x_n) \rangle ds. \end{aligned}$$

Since  $A^* Df : H \rightarrow H$  is bounded and continuous, the mapping:  $x \mapsto \mathcal{L}_0(P_t f)(x)$  is continuous, for any  $t \geq 0$ . Hence we can pass to the limit as  $n \rightarrow \infty$  in the previous identity and get

$$P_t f(x_0) - f(x_0) = \int_0^t \mathcal{L}_1(P_s f)(x_0) ds + \int_0^t \langle e^{sA} x_0, P_s (A^* Df)(x_0) \rangle ds.$$

This shows that (5.5) holds for  $x = x_0$ . □

To prove uniqueness for the Cauchy problem we need Itô's formula for OU processes and functions  $f \in C_A^2(H)$ . A special case of this formula which is enough for our purposes follows from [SZ11].

**Lemma 5.3.** *Let us consider an OU process  $(X_t^x)$  as in (4.3) and assume that the Lévy measure  $\nu$  has bounded support in  $H$ . Let  $f \in C_A^2(H)$ , we have (cf. (5.1))*

$$\begin{aligned} f(X_t^x) - f(x) &= \int_0^t \int_H [f(X_{s-}^x + y) - f(X_{s-}^x)] \tilde{N}(ds, dy) + \int_0^t \langle Df(X_r^x), dW_r^Q \rangle \\ &\quad + \int_0^t \mathcal{L}_1 f(X_r^x) dr + \int_0^t \langle A^* Df(X_r^x), X_r^x \rangle dr, \quad t \geq 0, x \in H. \end{aligned}$$

*Proof.* The assertion can be deduced by [SZ11, Lemma 7.4] using the decomposition formula (4.2). Note that it is a generalization of the finite-dimensional Itô's formula proved in [Ap09, Section 4.4].  $\square$

**Remark 5.4.** The assumption that  $\nu$  has bounded support can be removed in Lemma 5.3. The proof of this more general Itô's formula is not difficult but it requires many details (the basic idea is to use the Yosida approximations of  $A$  as in the proof of [SZ11, Lemma 7.4]).

Next we obtain well-posedness for the Cauchy problem (5.2).

**Theorem 5.5.** *Let us consider the Cauchy problem (5.2) with  $f \in C_A^2(H)$ . If we set  $u(t, x) = P_t f(x)$ ,  $t \geq 0$ ,  $x \in H$ , where  $(P_t)$  is defined in (4.4), then  $u$  is the unique bounded classical solution to (5.2).*

*Proof. Existence.* We know that  $u(t, x) = P_t f(x)$  is bounded and continuous on  $[0, +\infty) \times H$  (see (4.6)). Moreover, differentiating under the integral sign, it is straightforward to check that  $u(t, \cdot) \in C_A^2(H)$ ,  $t \geq 0$ , and  $D_x u$ ,  $A^* D_x u$ ,  $D_x^2 u$  are bounded and continuous functions on  $[0, +\infty) \times H$ .

From Theorem 5.1 we deduce that  $u(\cdot, x)$  is a  $C^1$ -function on  $[0, +\infty)$ , for any  $x \in H$ , and finally that  $u$  solves (5.2).

*Uniqueness.* We use Itô's formula as in the proof of Theorem 3.3.

For any  $n \geq 2$  we define Lévy processes  $U^n = (U_t^n)$ , and  $Z^n = (Z_t^n)$ , where  $Z_t^n = Z_t - U_t^n$  and

$$U_t^n = \int_0^t ds \int_{\{|x| > n\}} x N(ds, dx), \quad t \geq 0$$

(cf. (4.2)). Now  $U^n$  and  $Z^n$  are independent Lévy processes according to [PZ07, Lemma 4.24] (moreover,  $U^n$  is a compound Poisson process). By [PZ07, Lemma 4.25] it is straightforward to prove that

$$\mathbb{E}[e^{i\langle Z_t^n, h \rangle}] = \exp(-t\psi_n(h)), \quad h \in H, \quad (5.11)$$

where  $\psi_n : H \rightarrow \mathbb{C}$  is given as follows:

$$\psi_n(h) = \frac{1}{2} \langle Qh, h \rangle - i \langle a, h \rangle - \int_H \left( e^{i\langle h, y \rangle} - 1 - i \langle h, y \rangle \mathbf{1}_{\{|y| \leq 1\}}(y) \right) \nu_n(dy),$$

$h \in H$  (cf. (4.1)); here  $\nu_n(A) = \nu(A \cap \{|y| \leq n\})$ , for any Borel set  $A \subset H$  (i.e.,  $\nu_n(dy) = \mathbf{1}_{\{|y| \leq n\}}(y) \nu(dy)$ ). In particular,  $\nu_n$  has bounded support.

Let us fix  $x \in H$  and  $t > 0$ . According to page 84 in [Ap09] or [Ch87, Section 1] the OU process driven by  $Z^n$  is given by

$$\begin{aligned} X_t^n &= e^{tA}x + \int_0^t e^{(t-s)A} dZ_s^n = e^{tA}x + Y_t - \int_0^t e^{(t-s)A} dU_s^n \\ &= X_t^x - \int_0^t ds \int_{\{|x| > n\}} e^{(t-s)A} x N(ds, dx) = X_t^x - \sum_{0 < s \leq t} e^{(t-s)A} (\Delta Z_s) \mathbf{1}_{\{|\Delta Z_s| > n\}}, \end{aligned} \quad (5.12)$$

where  $\Delta Z_s = Z_s - Z_{s-}$  ( $Z_{s-}$  is the left limit in  $s$ ) and the last term is a finite random sum. Clearly, for any  $\omega \in \Omega$ ,  $(\int_0^t e^{(t-s)A} dU_s^n)(\omega) \rightarrow 0$  as  $n \rightarrow \infty$ . It follows that

$$X_t^n \rightarrow X_t^x \text{ as } n \rightarrow \infty, \quad \mathbb{P}\text{-a.s.} \quad (5.13)$$

Now we apply Itô's formula as in Lemma 5.3 to  $v(s, X_s^n)$ ,  $s \in [0, t]$ , where  $v(s, x) = u(t-s, x)$ ,  $s \in [0, t]$ ,  $x \in H$  (in the sequel we denote by  $\tilde{N}_n$  the compensated Poisson random measure of  $Z^n$ ). We find

$$\begin{aligned} v(t, X_t^n) - v(0, x) &= f(X_t^n) - u(t, x) \\ &= \int_0^t \int_H [u(t-r, X_{r-}^n + y) - u(t-r, X_{r-}^n)] \tilde{N}_n(dr, dy) \\ &\quad + \int_0^t \langle Du(t-r, X_r^n), dW_r^Q \rangle + \int_0^t (-\partial_s u(t-r, X_r^n) + \mathcal{L}_0 u(t-r, X_r^n)) dr \\ &\quad - \int_0^t dr \int_{\{|y| > n\}} [u(t-r, X_r^n + y) - u(t-r, X_r^n)] \nu(dy) \end{aligned}$$

(recall that the Lévy measure of  $Z^n$  is  $\nu_n(dy) = \mathbf{1}_{\{|y| \leq n\}}(y) \nu(dy)$ ). Since  $\partial_s u - \mathcal{L}_0 u = 0$ , by taking the expectation, we arrive at

$$\mathbb{E}[f(X_t^n)] = u(t, x) - \mathbb{E} \int_0^t dr \int_{\{|y| > n\}} [u(t-r, X_r^n + y) - u(t-r, X_r^n)] \nu(dy). \quad (5.14)$$

Note that

$$\left| \int_0^t dr \int_{\{|y| > n\}} [u(t-r, X_r^n + y) - u(t-r, X_r^n)] \nu(dy) \right| \leq 2t \|u\|_0 \nu(\{|y| > n\})$$

which tends to 0 as  $n \rightarrow \infty$ . Passing to the limit in (5.14) we get  $u(t, x) = \mathbb{E}[f(X_t^x)] = P_t f(x)$  (see also (5.13)). This proves the uniqueness.  $\square$

## 5.1 The Ornstein-Uhlenbeck generator $\mathcal{L}$ in $C_b(H)$

The contraction OU semigroup  $(P_t)$  acting on  $C_b(H)$  (i.e.,  $P_t : C_b(H) \rightarrow C_b(H)$ ,  $t \geq 0$ , and  $\|P_t f\|_0 \leq \|f\|_0$ ,  $t \geq 0$ ,  $f \in C_b(H)$ ) preserves the  $\pi$ -convergence (i.e., for any  $t \geq 0$ ,  $f_n \xrightarrow{\pi} f$  implies  $P_t f_n \xrightarrow{\pi} P_t f$ ) and further, for any  $f \in C_b(H)$ ,  $x \in H$ , the mapping:  $t \mapsto P_t f(x)$  is continuous from  $[0, +\infty)$  into  $\mathbb{R}$ .

Thus  $(P_t)$  belongs to the class of  $\pi$ -semigroups of contractions on  $C_b(H)$  considered in [Pr99]. Note that [Pr99] mainly deals with  $\pi$ -semigroups on  $UC_b(H)$ ; however according to [Pr99, Section 5] all the results in [Pr99] can be easily extended to  $\pi$ -semigroups on  $C_b(H)$ . Clearly  $(P_t)$  is also a *stochastically continuous Markov semigroup* according to the definition given in [Ma08]. Indeed the OU semigroup is given by kernels of probability measures.

As in [Pr99] and [Ma08] we define the *generator*  $\mathcal{L} : D(\mathcal{L}) \subset C_b(H) \rightarrow C_b(H)$  for  $(P_t)$  (we set  $\Delta_h = \frac{P_h - I}{h}$ ):

$$\left\{ \begin{array}{l} D(\mathcal{L}) = \{f \in C_b(H) \text{ such that } \exists g \in C_b(H), \lim_{h \rightarrow 0^+} \frac{P_h f(x) - f(x)}{h} = g(x), \\ \text{for any } x \in H, \text{ and } \sup_{h > 0} \|P_h f - f\|_0 h^{-1} < \infty\}; \\ \mathcal{L}f(x) = \lim_{h \rightarrow 0^+} \Delta_h f(x) = g(x), \quad f \in D(\mathcal{L}), x \in H. \end{array} \right. \quad (5.15)$$

**Remark 5.6.** One can give equivalent definitions for the generator of the OU semigroup. According to [Pr99, Proposition 3.6] the operator  $\mathcal{L}$  in (5.15) coincides with the generator considered in [Ce94] and [CG95] and defined by the Laplace transform of the semigroup. Moreover, by (4.7) one can apply [Pr99, Theorem 1.1] and obtain

$$\left\{ \begin{array}{l} D(\mathcal{L}) = \{f \in C_b(H) : \exists g \in C_b(H), \lim_{h \rightarrow 0^+} \sup_{x \in K} \left| \frac{P_h f(x) - f(x)}{h} - g(x) \right| = 0, \\ \text{for any compact set } K \subset H, \text{ and } \sup_{h > 0} \|P_h f - f\|_0 h^{-1} < \infty\}; \\ \mathcal{L}f(x) = \lim_{h \rightarrow 0^+} \Delta_h f(x) = g(x), \quad f \in D(\mathcal{L}), x \in H. \end{array} \right.$$

This shows that  $\mathcal{L} : D(\mathcal{L}) \subset C_b(H) \rightarrow C_b(H)$  coincides with the generator considered in [GK01], [Ku03] and [Ap07]. In particular [GK01] and [Ap07] use the mixed topology  $\tau$  on  $C_b(H)$ . This is the finest locally convex topology on  $C_b(H)$  which agrees on sup-norm bounded sets with the topology of the uniform convergence on compacts. In [Ap07] several properties of the OU semigroup  $(P_t)$  acting on  $(C_b(H), \tau)$  are established.

## 5.2 A $\pi$ -core for the generator $\mathcal{L}$

Generalizing (4.8) we say that a  $m$ -indexed multisequence  $(f_{n_1, \dots, n_m})_{n_1, \dots, n_m \in \mathbb{N}} \subset C_b(H)$   $\pi$ -converges to  $f \in C_b(H)$  if for any  $i = 1, \dots, m-1$  there exists an  $i$ -indexed multisequence  $(f_{n_1, \dots, n_i})_{n_1, \dots, n_i \in \mathbb{N}} \subset C_b(H)$  such that, for  $n_1, \dots, n_i \in \mathbb{N}$ ,

$$f_{n_1, \dots, n_{i+1}} \xrightarrow{\pi} f_{n_1, \dots, n_i} \text{ as } n_{i+1} \rightarrow \infty. \quad (5.16)$$

We write  $\lim_{n_1 \rightarrow \infty} \dots \lim_{n_m \rightarrow \infty} f_{n_1, \dots, n_m} \stackrel{\pi}{=} f$  or  $f_{n_1, \dots, n_m} \xrightarrow{\pi} f$ . Following [Ma08] we say that  $E \subset C_b(H)$  is  $\pi$ -dense if for any  $f \in C_b(H)$  there exists an  $m$ -indexed multisequence  $(f_{n_1, \dots, n_m})_{n_1, \dots, n_m \in \mathbb{N}} \subset E$  such that  $f_{n_1, \dots, n_m} \xrightarrow{\pi} f$ .

Moreover a subspace  $\mathcal{D} \subset D(\mathcal{L})$  is a  $\pi$ -core for  $\mathcal{L}$  if  $\mathcal{D}$  is  $\pi$ -dense in  $C_b(H)$  and, for any  $f \in D(\mathcal{L})$  there exists an  $m$ -indexed multisequence  $(f_{n_1, \dots, n_m})_{n_1, \dots, n_m \in \mathbb{N}} \subset \mathcal{D}$  such that

$$f_{n_1, \dots, n_m} \xrightarrow{\pi} f, \quad \mathcal{L}f_{n_1, \dots, n_m} \xrightarrow{\pi} \mathcal{L}f.$$

The next result can be proved more generally for any stochastically continuous Markov semigroup acting on  $C_b(H)$  (cf. [Ma08, Proposition 2.11]). It generalizes a classical result in the theory of  $C_0$ -semigroups.

**Theorem 5.7.** *Let  $(P_t)$  be the OU semigroup with generator  $\mathcal{L}$ . If a subspace  $\mathcal{D} \subset D(\mathcal{L})$  is  $\pi$ -dense in  $C_b(H)$  and moreover  $P_t(\mathcal{D}) \subset \mathcal{D}$ ,  $t \geq 0$ , then  $\mathcal{D}$  is a  $\pi$ -core for  $\mathcal{L}$ .*

As in [GK01] and [Ap07] and similarly to the space  $\mathcal{D}_0$ , we introduce

$$\tilde{\mathcal{D}}_0 = \{f \in C_A^2(H) : x \mapsto \langle x, A^* Df(x) \rangle \in C_b(H)\}. \quad (5.17)$$

Note that if  $f \in \tilde{\mathcal{D}}_0$  then  $\mathcal{L}_0 f \in C_b(H)$ . Using also Theorem 5.5, we prove that  $\tilde{\mathcal{D}}_0$  is an invariant  $\pi$ -core for the OU semigroup. We start with a preliminary result.

**Proposition 5.8.** *Let us consider the OU generator  $\mathcal{L}$  given in (5.15) and the operator  $\mathcal{L}_0$  defined in (5.4). The following statements hold:*

- (i)  $\tilde{\mathcal{D}}_0 \subset D(\mathcal{L})$  and  $\mathcal{L}f = \mathcal{L}_0 f$ , for any  $f \in \tilde{\mathcal{D}}_0$ ;
- (ii)  $\tilde{\mathcal{D}}_0$  is invariant for  $(P_t)$ , i.e.,  $P_t(\tilde{\mathcal{D}}_0) \subset \tilde{\mathcal{D}}_0$ ,  $t \geq 0$ .

*Proof.* (i) This assertion follows from [Ap07, Theorem 4.2] (see also Remark 5.6). Moreover, by (i) we deduce (see [Ap07] or [Pr99, Proposition 3.2])

$$P_t f(x) = f(x) + \int_0^t P_r \mathcal{L}_0 f(x) dr, \quad x \in H. \quad (5.18)$$

Alternatively, to prove (i) one can first establish (5.18) using Lemma 5.3 and arguing as in the final part of the proof of Theorem 5.5. Indeed we have, for  $f \in \tilde{\mathcal{D}}_0$ ,  $t > 0$ ,  $x \in H$ ,

$$\mathbb{E}[f(X_t^n)] - f(x) = \int_0^t \mathbb{E}[\mathcal{L}_0 f(X_r^n)] dr - \int_0^t dr \int_{\{|y|>n\}} \mathbb{E}[f(X_r^n + y) - f(X_r^n)] \nu(dy) \quad (5.19)$$

(the process  $(X_t^n)$  is defined in (5.12),  $n \geq 1$ ). Since  $\mathcal{L}_0 f \in C_b(H)$  we have  $\mathbb{E}[\mathcal{L}_0 f(X_r^n)] \rightarrow \mathbb{E}[\mathcal{L}_0 f(X_r^x)]$ , as  $n \rightarrow \infty$ ,  $r \in [0, t]$  (cf. (5.13)). Passing to the limit in (5.19) as  $n \rightarrow \infty$  we obtain (5.18). Once (5.18) is proved one can proceed as in the proof of Theorem 3.4 (see (3.17)) and get (i).

(ii) Differentiating under the integral sign we find  $P_t(C_b^2(H)) \subset C_b^2(H)$ ,  $t \geq 0$ . Moreover, we have easily  $A^* D P_t f \in C_b(H, H)$ , for  $f \in \tilde{\mathcal{D}}_0$ ,  $t \geq 0$ .

Thus, to prove (ii), it is enough to show that for  $f \in \tilde{\mathcal{D}}_0$ , we have that the map:

$$x \mapsto \langle x, A^* D P_t f(x) \rangle \in C_b(H), \quad t \geq 0. \quad (5.20)$$

To check this we use Theorem 5.1 and (5.18). We obtain for  $f \in \tilde{\mathcal{D}}_0$ ,  $x \in H$ ,  $t \geq 0$ ,

$$\int_0^t P_s \mathcal{L}_0 f(x) ds = \int_0^t \mathcal{L}_0 P_s f(x) ds.$$

Since  $s \mapsto \mathcal{L}_0 P_s f(x)$  and  $s \mapsto P_s \mathcal{L}_0 f(x)$  are both continuous functions we get

$$\mathcal{L}_0 P_t f = P_t \mathcal{L}_0 f, \quad t \geq 0.$$

Let us fix  $t > 0$ . The previous identity shows that  $\mathcal{L}_0 P_t f \in C_b(H)$  since  $\mathcal{L}_0 f \in C_b(H)$ . We have (see (5.4))  $\mathcal{L}_0 P_t f(x) = \mathcal{L}_1 P_t f(x) + \langle x, A^* D P_t f(x) \rangle$ ,  $x \in H$ . Since  $\mathcal{L}_1 P_t f \in C_b(H)$  it follows that  $\langle \cdot, A^* D P_t f(\cdot) \rangle \in C_b(H)$ ,  $t \geq 0$ .  $\square$

We need to introduce a space  $\mathcal{D}_1$  which is similar to the space  $I_A(H)$  used in [Ma08] (see also Remark 2.25 in [Da04]).

$\mathcal{D}_1$  is the linear span of the real and imaginary parts of the maps  $\phi_{a,h} : H \rightarrow \mathbb{C}$ ,

$$x \mapsto \phi_{a,h}(x) = \int_0^a P_s(e^{i\langle \cdot, h \rangle})(x) ds = \int_0^a e^{i\langle e^{sA} x, h \rangle} e^{-\int_0^s \psi(e^{rA^*} h) dr} ds, \quad (5.21)$$

where  $h \in D(A^*)$ ,  $a > 0$  (cf. (4.5)).

**Proposition 5.9.** *Let  $\mathcal{L}$  be the OU generator. The following statements hold:*

- (i)  $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_0 \subset D(\mathcal{L})$  and  $\mathcal{L}f = \mathcal{L}_0 f$ ,  $f \in \mathcal{D}_1$ ;
- (ii)  $\mathcal{D}_1$  is invariant for the OU semigroup  $(P_t)$ , i.e.,  $P_t(\mathcal{D}_1) \subset \mathcal{D}_1$ ,  $t \geq 0$ ;
- (iii) the space  $\mathcal{D}_1$  is  $\pi$ -dense in  $C_b(H)$ .

*Proof.* (i) By Proposition 5.8 we have only to prove that  $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_0$ . If  $I_1(\phi_{a,h})$  and  $I_2(\phi_{a,h})$  denote respectively the real and imaginary part of  $\phi_{a,h}$ , it is enough to show that  $I_j(\phi_{a,h}) \in \mathcal{D}_1$ ,  $j = 1, 2$ .

We have first to prove that  $I_j(\phi_{a,h}) \in C_A^2(H)$  (see (5.17)). If we fix  $a > 0$  and  $h \in D(A^*)$  we can compute, for  $x \in H$ ,  $k \in H$ ,

$$\langle D\phi_{a,h}(x), k \rangle = \int_0^a \langle D[P_s(e^{i\langle \cdot, h \rangle})](x), k \rangle ds = i \int_0^a \langle e^{sA^*} h, k \rangle P_s(e^{i\langle \cdot, h \rangle})(x) ds;$$

it follows that  $DI_j(\phi_{a,h}) \in C_b(H, H)$ ,  $j = 1, 2$ . Moreover,  $DI_j(\phi_{a,h}(x)) \in D(A^*)$ ,  $x \in H$ , and  $A^* D\phi_{a,h}(x) = i \int_0^a e^{sA^*} A^* h P_s(e^{i\langle \cdot, h \rangle})(x) ds$ ; we deduce that

$$A^* DI_j(\phi_{a,h}) \in C_b(H, H), \quad j = 1, 2.$$

Since

$$\langle D^2\phi_{a,h}(x)k', k \rangle = - \int_0^a \langle e^{sA^*} h, k \rangle \langle e^{sA^*} h, k' \rangle P_s(e^{i\langle \cdot, h \rangle})(x) ds, \quad k, k' \in H,$$

using an orthonormal basis  $(e_k)$ , we find that  $D^2 I_j(\phi_{a,h})(x)$  is a Hilbert-Schmidt operator for  $j = 1, 2$ ,  $x \in H$ . Moreover, for  $x, y \in H$ ,

$$\begin{aligned} & \|D^2 I_j(\phi_{a,h})(x) - D^2 I_j(\phi_{a,h})(y)\|_{HS}^2 \\ & \leq C_a \int_0^a |h|^2 |e^{sA}(x - y)|^2 \sum_{j,k=1}^{\infty} (\langle e^{sA^*} h, e_k \rangle \langle e^{sA^*} h, e_j \rangle)^2 ds \\ & \leq c_a |h|^2 |x - y|^2 \int_0^a |e^{sA^*} h|^4 ds. \end{aligned}$$

Using also the previous formula we obtain that  $D^2 I_j(\phi_{a,h}) : H \rightarrow L_2(H)$  is bounded and continuous,  $j = 1, 2$ . This shows that  $I_j(\phi_{a,h}) \in C_A^2(H)$ .



To finish the proof it remains to prove that  $x \mapsto \langle x, A^* DI_j(\phi_{a,h})(x) \rangle \in C_b(H)$ ,  $j = 1, 2$ . We have, integrating by parts,

$$\begin{aligned} \langle A^* D\phi_{a,h}(x), x \rangle &= i \int_0^a \langle e^{sA^*} A^* h, x \rangle P_s(e^{i\langle \cdot, h \rangle})(x) ds \\ &= i \int_0^a \langle e^{sA^*} A^* h, x \rangle e^{i\langle x, e^{sA^*} h \rangle} e^{-\int_0^s \psi(e^{rA^*} h) dr} ds = \int_0^a \frac{d}{ds} [e^{i\langle x, e^{sA^*} h \rangle}] e^{-\int_0^s \psi(e^{rA^*} h) dr} ds \\ &= e^{i\langle e^{aA^*} x, h \rangle} e^{-\int_0^a \psi(e^{rA^*} h) dr} - e^{i\langle x, h \rangle} + \int_0^a e^{i\langle e^{sA^*} x, h \rangle} e^{-\int_0^s \psi(e^{rA^*} h) dr} \psi(e^{sA^*} h) ds. \end{aligned}$$

This shows that  $x \mapsto \langle A^* D\phi_{a,h}(x), x \rangle$  is a bounded and continuous function. The assertion follows easily.

(ii) Let us fix  $t > 0$ . We prove that  $P_t(\mathcal{D}_1) \subset \mathcal{D}_1$ . We have, for  $a > 0$ ,  $x \in H$ ,  $h \in D(A^*)$ , by using the semigroup law,

$$P_t \phi_{a,h}(x) = \int_0^a P_{s+t}(e^{i\langle \cdot, h \rangle})(x) ds = \int_0^{a+t} P_r(e^{i\langle \cdot, h \rangle})(x) dr - \int_0^t P_r(e^{i\langle \cdot, h \rangle})(x) dr. \quad (5.22)$$

Hence  $P_t \phi_{a,h}(x) = \phi_{a+t,h}(x) - \phi_{t,h}(x)$ . This shows the assertion.

(iii) One can follow the proof of [Ma08, Proposition 4.4]. This is based on two facts. The first one is that the linear span of all real and imaginary parts of functions  $e^{i\langle \cdot, h \rangle}$ ,  $h \in D(A^*)$ , is  $\pi$ -dense in  $C_b(H)$  (see [Da04, Proposition 2.37] or [Ma08, Proposition 4.2]). The second fact is that  $n \phi_{\frac{1}{n},h}(x)$  converges to  $e^{i\langle x, h \rangle}$  as  $n \rightarrow \infty$ ,  $x \in H$ ,  $h \in D(A^*)$ .  $\square$

Since by Propositions 5.8 and 5.9 the spaces  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  both satisfy the assumptions of Proposition 5.7 we obtain

**Corollary 5.10.** *The spaces  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  are both  $\pi$ -cores for the OU generator  $\mathcal{L}$  with  $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_0$ . They are also invariant for the OU semigroup  $(P_t)$  and, for any  $f \in \tilde{\mathcal{D}}_0$ , one has  $\mathcal{L}f = \mathcal{L}_0 f$ .*

**Remark 5.11.** In [GK01, Theorem 4.5] and [Ap07, Theorem 5.2] the authors consider the space  $\mathcal{FC}_A^2(H) \subset C_b^2(H)$  of all cylindrical functions  $f : H \rightarrow \mathbb{R}$  such that there exists  $n \in \mathbb{N}$ ,  $h_1, \dots, h_n \in D(A^*)$  and  $g \in C_b^2(\mathbb{R}^n)$  with

$$f(x) = g(\langle x, h_1 \rangle, \dots, \langle x, h_n \rangle), \quad x \in H, \quad (5.23)$$

and moreover the map:  $x \mapsto \langle A^* Df(x), x \rangle \in C_b(H)$ . In [GK01] and in [Ap07] it is stated that  $\mathcal{FC}_A^2(H)$  is a core for the OU generator  $\mathcal{L}$  with respect to the mixed topology (cf. Remark 5.6); for this result [GK01] considers the case of Gaussian OU processes and [Ap07] assumes that  $\int_{\{|y|>1\}} |y| \nu(dy) < \infty$ . However it seems that this result requires additional assumptions on  $A$ . To this purpose we only note that even if  $g \in C_K^2(\mathbb{R})$  and  $h \in D(A^*)$ , then in general the cylindrical function  $f(x) = g(\langle x, h \rangle)$ ,  $x \in H$ , does not belong to  $\mathcal{FC}_A^2(H)$ . As in [LR02] a sufficient condition in order that  $\mathcal{FC}_A^2(H)$  is a core is that there exists in  $H$  an orthonormal basis  $\mathcal{B}$  of eigenvectors of  $A^*$ ; in this case  $\mathcal{FC}_A^2(H)$  is a core if in the previous definition we add the condition that the vectors  $h_1, \dots, h_n$  in (5.23) belong to  $\mathcal{B}$  (cf. the proof of Theorem 4.5 in [GK01]).

**Remark 5.12.** As an extension of the OU semigroups one can consider the generalised Mehler semigroups (see [BRS96], [FR00], [PZ06], [We13]). A generalised Mehler semigroup  $(S_t)$ , acting on  $C_b(H)$ , is given by

$$S_t f(x) = \int_H f(e^{tA}x + y) \mu_t(dy), \quad t \geq 0, x \in H, f \in C_b(H),$$

where  $(e^{tA})$  is a  $C_0$ -semigroup on  $H$ , with generator  $A$ ,  $\mu_t$ ,  $t \geq 0$ , is a given family of probability measures on  $H$ , such that  $\hat{\mu}_t(h) = \exp\left(-\int_0^t \psi(e^{sA^*}h) ds\right)$ ,  $h \in H$ ,  $t \geq 0$ . Here,  $\psi : H \rightarrow \mathbb{C}$  is a (norm) continuous, negative definite function with  $\psi(0) = 0$ . Moreover, we require that  $t \mapsto \mu_t$  is continuous on  $[0, +\infty)$  with respect to the weak topology of measures (cf. Lemma 2.1 in [BRS96]).

One can define a generator  $\mathcal{L}$  for  $(S_t)$  with  $D(\mathcal{L}) \subset C_b(H)$  as in (5.15) and an associated subspace  $\mathcal{D}_1$  (defined with  $A$  and  $\psi$  given before). Arguing as in [Ma08, Proposition 4.4] and using (5.22) one shows that  $\mathcal{D}_1 \subset D(\mathcal{L})$  and

$$\mathcal{L}\phi_{a,h}(x) = e^{i\langle e^{aA}x, h \rangle} e^{-\int_0^a \psi(e^{rA^*}h) dr} - e^{i\langle x, h \rangle}, \quad x \in H, h \in D(A^*), a > 0.$$

Moreover,  $\mathcal{D}_1$  is an invariant  $\pi$ -core for  $(S_t)$ . To prove this fact first we argue as in the proof of Proposition 5.9 and establish that  $S_t(\mathcal{D}_1) \subset \mathcal{D}_1$ ,  $t \geq 0$ ; then we apply Proposition 5.7 to  $(S_t)$  and  $\mathcal{D}_1$ .

## 6 Kolmogorov equations for measures

Following [Ma08] the spaces  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  (see Section 5.2) can be used to characterize the marginal distributions of the Ornstein-Uhlenbeck process as solutions to Fokker-Planck-Kolmogorov equations for measures. Note that [Ma08] deals with stochastically continuous Markov semigroups  $(R_t)$  acting on  $UC_b(H)$  (i.e.,  $R_t : UC_b(H) \rightarrow UC_b(H)$ ,  $t \geq 0$ ). However all the results in [Ma08] (see in particular Theorems 1.2, 1.3 and 1.4) can be easily proved for stochastically continuous Markov semigroups acting on  $C_b(H)$  with the same proofs.

To state the main result let  $M(H)$  be the Banach space of all finite signed Borel measures on  $H$  endowed with the total variation norm  $\|\cdot\|_{TV}$ .

**Definition 6.1.** Let  $\mathcal{D}$  be  $\tilde{\mathcal{D}}_0$  or  $\mathcal{D}_1$  and  $\mathcal{L}_0$  be the operator defined in (5.4). Given  $\mu \in M(H)$ , a family of measures  $(\gamma_t)_{t \geq 0} \subset M(H)$  is called a *solution to the measure equation*

$$\begin{cases} \frac{d}{dt} \int_H f(x) \gamma_t(dx) = \int_H \mathcal{L}_0 f(x) \gamma_t(dx), & f \in \mathcal{D}, t \geq 0, \\ \gamma_0 = \mu \end{cases} \quad (6.1)$$

if we have:

- (i) for any  $T > 0$ , the real map:  $t \mapsto \|\gamma_t\|_{TV}$  belongs to  $L^1(0, T)$ ;
- (ii) for any  $f \in \mathcal{D}$ , the real function:  $t \mapsto \int_H f(x) \gamma_t(dx)$  is absolutely continuous on each  $[0, T]$ ,  $T > 0$ , and moreover

$$\int_H f(x) \gamma_t(dx) - \int_H f(x) \mu(dx) = \int_0^t \left( \int_H \mathcal{L}_0 f(x) \gamma_s(dx) \right) ds, \quad t \geq 0. \quad \square$$

To study (6.1) one associates by duality to the OU semigroup  $(P_t)$  another semigroup  $(P_t^*)$  (see [Ma08, Section 3]);  $P_t^* : M(H) \rightarrow M(H)$ ,  $t \geq 0$ , and, for any  $\mu \in M(H)$ ,

$$P_t^* \mu(B) = \int_H P_t(\mathbf{1}_B)(x) \mu(dx),$$

for Borel set  $B \subset H$ .

**Theorem 6.2.** *Let  $\mathcal{D}$  be  $\tilde{\mathcal{D}}_0$  or  $\mathcal{D}_1$  (see Section 5.2). Then for any  $\mu \in M(H)$  there exists a unique solution  $(\gamma_t)_{t \geq 0} \subset M(H)$  to equation (6.1) Moreover, such solution is given by  $(P_t^* \mu)_{t \geq 0}$ .*

*Proof.* The assertion follows from [Ma08, Theorem 1.5] (see also [Ma08, Remark 5.1]) using the fact that both  $\tilde{\mathcal{D}}_0$  and  $\mathcal{D}_1$  are  $\pi$ -cores for the OU generator  $\mathcal{L}$  (see Corollary 5.10).  $\square$

## Appendix

**Sketch of the proof of Lemma 3.2.** Let  $f \in C_b^2(\mathbb{R}^d)$ . We proceed in some steps.

*I Step.* We consider a  $C^\infty$ -function  $\rho : \mathbb{R}^d \rightarrow \mathbb{R}$  such that, for all  $x \in \mathbb{R}^d$ ,  $0 \leq \rho(x) \leq 1$ ,  $\rho(x) = 1$  for  $|x| \leq 1$  and  $\rho(x) = 0$  for  $|x| \geq 3/2$ . We define a standard sequence of mollifiers  $(\rho_n)$  setting  $\rho_n(x) = \frac{1}{c} \rho(nx) n^d$ , where  $c = \int_{\mathbb{R}^d} \rho(x) dx$ . Therefore,  $\int_{\mathbb{R}^d} \rho_n(x) dx = 1$ ,  $n \geq 1$ .

We introduce a sequence  $(\tilde{f}_n)$  as  $\tilde{f}_n(x) = (f * \rho_n)(x) = \int_{\mathbb{R}^d} f(y) \rho_n(x - y) dy$ ,  $x \in \mathbb{R}^d$ . Differentiating under the integral sign, one proves easily that each  $\tilde{f}_n$  belongs to  $C_b^\infty(\mathbb{R}^d)$  (i.e., each  $\tilde{f}_n$  is bounded and has bounded derivatives of all orders). It is straightforward to check that, for any compact  $K \subset \mathbb{R}^d$ ,

$$\tilde{f}_n \rightarrow f, \quad D\tilde{f}_n \rightarrow Df, \quad D^2\tilde{f}_n \rightarrow D^2f,$$

uniformly on  $K$  as  $n \rightarrow \infty$ . Moreover, for any  $n \geq 1$ ,

$$\|\tilde{f}_n\|_0 + \|D\tilde{f}_n\|_0 + \|D^2\tilde{f}_n\|_0 \leq \|f\|_0 + \|Df\|_0 + \|D^2f\|_0. \quad (6.2)$$

*II Step.* We define  $f_n^* : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $n \geq 1$ ,

$$f_n^*(x) = \tilde{f}_n(x) \rho\left(\frac{x}{n}\right), \quad x \in \mathbb{R}^d;$$

each  $f_n^*$  belongs to  $C_b^\infty(\mathbb{R}^d)$  with compact support in the  $d$ -dimensional cube  $(-2n, 2n)^d$ . By (6.2) we obtain that  $(f_n^*)$ ,  $(Df_n^*)$  and  $(D^2f_n^*)$  are uniformly bounded on  $\mathbb{R}^d$ , i.e.,  $\sup_{n \geq 1} (\|f_n^*\|_0 + \|Df_n^*\|_0 + \|D^2f_n^*\|_0) < \infty$ . In addition, as  $n \rightarrow \infty$ , we have  $f_n^* \rightarrow f$ ,  $Df_n^* \rightarrow Df$ ,  $D^2f_n^* \rightarrow D^2f$  pointwise on  $\mathbb{R}^d$ .

*III Step.* We define suitable extensions  $(f_n)$  of the functions  $(f_n^*)$  when these are restricted to the domain  $[-2n, 2n]^d$ . Let  $n \geq 1$ . We extend  $f_n^*$  from  $[-2n, 2n]^d$  to  $\mathbb{R}^d$  by periodicity of period  $4n$  in all its variables, i.e.,

$$f_n(x + 4nh) = f_n^*(x), \quad x \in [-2n, 2n]^d, \quad h = (h_1, \dots, h_d) \in \mathbb{Z}^d.$$

Clearly,  $\sup_{n \geq 1} (\|f_n\|_0 + \|Df_n\|_0 + \|D^2f_n\|_0) < \infty$  and  $(f_n) \subset C_b^\infty(\mathbb{R}^d)$ . Since, for all  $n \geq 1$ ,  $f_n$  has period  $4n$  in each of its variables, it can be represented by Fourier series of the kind

$$f_n(x) = \sum_{h \in \mathbb{Z}^d} c_h^{(n)} e^{\frac{i\pi}{2n} \langle x, h \rangle}, \quad x \in \mathbb{R}^d, \quad (6.3)$$

with  $c_h^{(n)} = \frac{1}{(4n)^d} \int_{[-2n, 2n]^d} f_n(x) e^{-\frac{i\pi}{2n} \langle x, h \rangle} dx$  and  $\langle x, h \rangle = x_1 h_1 + \dots + x_d h_d$ . It is a standard result that the series is uniformly convergent on  $\mathbb{R}^d$  (see [SW71, Chapter VII]). We introduce, for  $n, m \geq 1$ ,

$$f_{nm}(x) = \sum_{h \in \mathbb{Z}^d, |h| \leq m} c_h^{(n)} e^{\frac{i\pi}{2n} \langle x, h \rangle} \in V(\mathbb{R}^d).$$

Differentiating under the summation in (6.3) (using the regularity properties of  $f_n$ ) we get that, for any  $n \geq 1$ ,  $\lim_{m \rightarrow \infty} f_{nm} = f_n$ ,  $\lim_{m \rightarrow \infty} Df_{nm} = Df_n$  and  $\lim_{m \rightarrow \infty} D^2f_{nm} = D^2f_n$ , uniformly on  $\mathbb{R}^d$ . This gives the first assertion of Lemma 3.2.

*IV Step.* Let  $x_0 \in \mathbb{R}^d$ . there exists  $n_0 \in \mathbb{N}$  such that  $|x_0| < n_0$ ; therefore, by Step III,  $f_n(x_0) = f_n^*(x_0)$ ,  $n \geq n_0$ . This implies by Step II that  $f_n(x_0) \rightarrow f(x_0)$  as  $n \rightarrow \infty$ . The same happens for the derivatives, that is  $Df_n(x_0) \rightarrow Df(x_0)$ ,  $D^2f_n(x_0) \rightarrow D^2f(x_0)$ , as  $n \rightarrow \infty$ . We have shown that  $(f_n)$  and  $(f_{nm})$  verify all the assertions.  $\square$

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